

# Occupation times of general Lévy processes

Lan Wu · Jiang Zhou · Shuang Yu

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**Abstract** For an arbitrary Lévy process  $X$  which is not a compound Poisson process, we are interested in its occupation times. We use a quite novel and useful approach to derive formulas for the Laplace transform of the joint distribution of  $X$  and its occupation times. Our formulas are compact, and more importantly, the forms of the formulas clearly demonstrate the essential quantities for the calculation of occupation times of  $X$ . It is believed that our results are important not only for the study of stochastic processes, but also for financial applications.

**Keywords** Occupation times · Lévy processes · Laplace transform · Infinitely divisible distribution · Strong Markov property · Continuity theorem

**Mathematics Subject Classification (2010)** MSC 60

## 1 Introduction

The investigation of occupation times of stochastic processes is an interesting and historic question. In 1939, Paul Lévy derived an interesting and useful result:

$$\mathbb{P}\left(\int_0^t \mathbf{1}_{\{W_u \geq 0\}} du \in ds\right) = \frac{ds}{\pi \sqrt{s(t-s)}}, \quad 0 < s < t, \quad (1.1)$$

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Lan Wu  
School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China  
E-mail: lwu@pku.edu.cn

Jiang Zhou  
School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China  
Tel.: +86 18810515809  
E-mail: 1101110056@pku.edu.cn

Shuang Yu  
School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China  
E-mail: 1101110054@pku.edu.cn

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion and  $\mathbf{1}_A$  is the indicator function of a set  $A$ ; see Lévy [13] for the details. After that, the investigation on occupation times of Lévy processes (in particular spectrally negative Lévy processes) has made much great progress. For example, the Laplace transform of  $\int_0^\infty \mathbf{1}_{\{X_t < 0\}} dt$  and the joint Laplace transform of  $\tau_0^-$  and  $\int_0^{\tau_0^-} \mathbf{1}_{\{a < X_t < b\}} dt$  have been derived, where  $X = (X_t)_{t \geq 0}$  is a spectrally negative Lévy process;  $\tau_0^-$  is the first passage time of  $X$  and  $0 \leq a \leq b$ . The interested readers are referred to [12, 15] for more details.

There are many papers considering the joint distribution of a Lévy process and its occupation times. For instance, for a spectrally negative Lévy process  $X$ , the Laplace transform of  $\mathbb{E}_x \left[ e^{-p \int_0^t \mathbf{1}_{\{a < X_s < b\}} ds} \mathbf{1}_{\{X_t \in dy\}} \right]$  with respect to  $t$  has been considered in [11]. Recently, Wu and Zhou [18] studied a similar problem, where the process  $X$  is assumed to be a hyper-exponential jump diffusion process. Here, we want to mention that Cai et al. [5] have derived formulas for

$$\int_0^\infty e^{-(a+r)t} \mathbb{E}_x \left[ e^{-p \int_0^t \mathbf{1}_{\{X_s \leq b\}} ds + \gamma X_t} \right] dt,$$

where  $X$  is a double exponential jump diffusion process.

The above mentioned papers can be classified into two categories according to the assumption on the process  $X$ . Some papers assume that the process  $X$  is a spectrally negative Lévy process (e.g., [11, 15]), the others allow the process  $X$  to have two-sided jumps but pose a limitation on its jumps (the jumps of  $X$  follow exponential or hyper-exponential distributions). These two categories both have some drawbacks. For the first category, the results in those papers are written in terms of  $q$ -scale functions, which are associated to spectrally negative Lévy processes; thus it is very difficult to extend their results and approaches to the case that the process  $X$  has both positive and negative jumps. For the second one, the derivation in these papers are heavily dependent on the assumption of exponential-type jump distributions; therefore, it is likely that their approaches cannot be used to other non-exponential-type jump distributions.

In this paper, for an arbitrary Lévy process  $X$  but not a compound Poisson process, we explore the problem how to compute the following quantity:

$$\mathbb{E}_x \left[ e^{-p \int_0^t \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_t \in dy\}} \right], \quad (1.2)$$

where  $p$  is an appropriate constant. Formulas for the Laplace transform of (1.2) with respect to  $t$  are derived by applying a novel but straightforward approach. Our method consists of two steps. First, we consider the case that  $X$  is a jump diffusion process with jumps having rational Laplace transform. Then the result is extended to a general Lévy process via an approximation discussion. As in [11, 18], the result in this article has some financial applications. Specifically, our results can be used in pricing occupation time derivatives. It is expected to obtain some unusual and profound outcomes on the pricing of occupation time derivatives through the application of the general result obtained in this paper. But here, we do not intend to discuss this application further and leave it to future research.

The remainder of the paper is organized as follows. Some important preliminary results related to Lévy processes are given in Section 2, and the main results are presented in Section 3. In the next two sections, details on the derivation are presented. Finally, we present two examples in Section 6 and draw some conclusions in Section 7.

## 2 Some preliminary results

In this paper, we let  $X = (X_t)_{t \geq 0}$  represent a general Lévy process. The law and the corresponding expectation of  $X$  such that  $X_0 = x$  are denoted respectively by  $\mathbb{P}_x$  and  $\mathbb{E}_x$ . To simplify the notation, we write  $\mathbb{P}$  and  $\mathbb{E}$  when  $x = 0$ . In addition, define  $\underline{X}_T := \inf_{0 \leq t \leq T} X_t$  and  $\overline{X}_T := \sup_{0 \leq t \leq T} X_t$  for  $T \geq 0$ , and denote

$$\int_a^b := \int_{(a,b)}, \quad \int_{a^-}^b := \int_{[a,b)} \quad \text{and} \quad \int_a^{b^+} := \int_{(a,b]}, \quad (2.1)$$

where  $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ .

Throughout this article, we assume that  $X$  is not a compound Poisson process; and the random variable  $e(q)$  for  $q > 0$ , independent of  $X$ , is an exponential random variable with rate  $q$ ;  $Re(x)$  and  $Im(x)$  represent the real part and the imaginary part of a complex number  $x$ , respectively.

The following lemma, which is taken from Proposition 15 on page 30 in Bertoin [3], is important for the derivation in the paper.

**Lemma 2.1** *For any  $q > 0$  and  $z \in \mathbb{R}$ , we have  $\mathbb{P}(X_{e(q)} = z) = 0$ , which leads to that  $\mathbb{P}(X_t = z) = 0$  for Lebesgue almost every  $t > 0$ .*

**Remark 2.1** *If  $X$  is a compound Poisson process, then it is possible that  $\mathbb{P}(X_{e(q)} = z) > 0$  for some  $z \in \mathbb{R}$ , which will make the discussion more difficult.*

The result in Lemma 2.2 is well-known, one can see, e.g., Theorem 5 on page 160 in Bertoin [3].

**Lemma 2.2** *For  $Re(\xi) \leq 0$  and  $q > 0$ , we have*

$$\mathbb{E} \left[ e^{\xi \overline{X}_{e(q)}} \right] = e^{\int_0^\infty \frac{1}{t} e^{-qt} \int_0^\infty (e^{\xi x} - 1) \mathbb{P}(X_t \in dx) dt}. \quad (2.2)$$

The following theorem gives some simple but useful results, and its derivation is straightforward.

**Theorem 2.1** (i) *For  $p, q > 0$ , there exists an infinitely divisible distribution  $G_1(x)$  on  $[0, \infty)$  with the Laplace transform*

$$\int_{0^-}^\infty e^{-sx} dG_1(x) = \frac{\mathbb{E} \left[ e^{-s \overline{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s \overline{X}_{e(p+q)}} \right]} = e^{\int_0^\infty (e^{-sx} - 1) \Pi_1(dx)}, \quad s \geq 0, \quad (2.3)$$

where  $\Pi_1(dx)$  is the Lévy measure and is given by

$$\Pi_1(dx) = \int_0^\infty \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t \in dx) dt, \quad x > 0. \quad (2.4)$$

(ii) For  $q > 0$  and  $-q < p < 0$ , there are two measures  $G_{21}(x)$  and  $G_{22}(x)$  on  $[0, \infty)$  such that

$$\int_{0-}^\infty e^{-sx} dG_{21}(x) = \frac{1}{2} \left( e^{-\int_0^\infty e^{-sx} \Pi_2(dx)} + e^{\int_0^\infty e^{-sx} \Pi_2(dx)} \right), \quad s > 0, \quad (2.5)$$

and

$$\int_{0-}^\infty e^{-sx} dG_{22}(x) = \frac{1}{2} \left( e^{\int_0^\infty e^{-sx} \Pi_2(dx)} - e^{-\int_0^\infty e^{-sx} \Pi_2(dx)} \right), \quad s > 0, \quad (2.6)$$

where

$$\Pi_2(dx) = \int_0^\infty \frac{1}{t} e^{-qt} (e^{-pt} - 1) \mathbb{P}(X_t \in dx) dt, \quad x > 0. \quad (2.7)$$

Besides, it holds that

$$\begin{aligned} & e^{\int_0^\infty \Pi_2(dx)} \int_{0-}^\infty e^{-sx} d(G_{21}(x) - G_{22}(x)) \\ &= \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+q)}} \right]} = e^{\int_0^\infty (1 - e^{-sx}) \Pi_2(dx)}, \quad s > 0, \end{aligned} \quad (2.8)$$

where

$$\int_0^\infty \Pi_2(dx) = \int_0^\infty \frac{1}{t} e^{-qt} (e^{-pt} - 1) \mathbb{P}(X_t > 0) dt < \infty. \quad (2.9)$$

(iii)  $G_1(x)$ ,  $G_{21}(x)$  and  $G_{22}(x)$  are continuous on  $(0, \infty)$ .

*Proof* (i) According to (2.2), we can derive

$$\frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+q)}} \right]} = e^{\int_0^\infty (e^{-sx} - 1) \Pi_1(dx)}, \quad \text{for } s \geq 0, \quad (2.10)$$

where  $\Pi_1(dx)$  is given by (2.4) and is a measure (since  $p > 0$ ). Note that

$$\Pi_1(0, \infty) := \int_0^\infty \Pi_1(dx) = \int_0^\infty \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t > 0) dt < \infty. \quad (2.11)$$

Therefore, from the Lévy-Khintchine formula (see, e.g., Theorem 8.1 on page 37 in Sato [16]), we obtain that the right-hand side of (2.10) is the Laplace transform of an infinitely divisible distribution, i.e., there is an infinitely divisible distribution  $G_1(x)$  on  $[0, \infty)$  such that

$$\int_{0-}^\infty e^{-sx} dG_1(x) = e^{\int_0^\infty (e^{-sx} - 1) \Pi_1(dx)}, \quad s \geq 0. \quad (2.12)$$

Formula (2.3) is derived from (2.10) and (2.12).

(ii) It follows from (2.2) that

$$\frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+q)}} \right]} = e^{\int_0^\infty (1-e^{-sx}) \Pi_2(dx)}, \quad s \geq 0, \quad (2.13)$$

where  $\Pi_2(dx)$  is given by (2.7) and is a measure (since  $p < 0$ ). Next, it is obvious that

$$e^{-\int_0^\infty e^{-sx} \Pi_2(dx)} = \sum_{n=0}^{\infty} \frac{(-1)^n \int_0^\infty e^{-sx} d\Pi_2^{*n}(0, x)}{n!}, \quad (2.14)$$

where  $d\Pi_2^{*0}(0, x) = \delta_0(dx)$ ;  $\Pi_2(0, x) := \int_0^x \Pi_2(dy)$  and  $\Pi_2^{*n}(0, x)$  for  $n \geq 1$  is the  $n$ -fold convolution of  $\Pi_2(0, x)$ .

Therefore, for  $x \geq 0$ , we can define

$$G_{21}(x) = 1 + \sum_{n=1}^{\infty} \frac{\Pi_2^{*2n}(0, x)}{(2n)!} \quad \text{and} \quad G_{22}(x) = \sum_{n=1}^{\infty} \frac{\Pi_2^{*(2n-1)}(0, x)}{(2n-1)!}, \quad (2.15)$$

which are measures on  $[0, \infty)$  obviously. Formulas (2.5) and (2.6) follow directly from (2.15), and formula (2.8) is due to (2.5), (2.6) and (2.13).

(iii) Formula (2.3) gives

$$\begin{aligned} \int_{0-}^{\infty} e^{-sx} dG_1(x) &= e^{\int_0^\infty (e^{-sx} - 1) \Pi_1(dx)} \\ &= e^{-\Pi_1(0, \infty)} \sum_{n=0}^{\infty} \frac{\int_0^\infty e^{-sx} d\Pi_1^{*n}(0, x)}{n!}, \end{aligned} \quad (2.16)$$

where  $d\Pi_1^{*0}(0, x) = \delta_0(dx)$ ;  $\Pi_1(0, x) := \int_0^x \Pi_1(dy)$  and  $\Pi_1^{*n}(0, x)$  for  $n \geq 1$  is the  $n$ -fold convolution of  $\Pi_1(0, x)$ . From (2.4) and Lemma 2.1, we know  $\Pi_1(dx)$  has no atoms. Thus,  $G_1(x)$  is continuous on  $(0, \infty)$ .

Since  $\Pi_2(dx)$  has no atoms, the conclusion that  $G_{21}(x)$  and  $G_{22}(x)$  are continuous on  $(0, \infty)$  can be seen from (2.15).  $\square$

**Remark 2.2** If we define  $G_1(x) = 0$  for  $x < 0$ , then from (2.3), we obtain that  $G_1(x)$  is not left-continuous at 0 since

$$G_1(0) = \lim_{s \uparrow \infty} \int_{0-}^{\infty} e^{-sx} dG_1(x) = e^{-\int_0^\infty \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t > 0) dt} > 0. \quad (2.17)$$

**Remark 2.3** From Theorem 2.1 (ii), it is easy to derive that

$$G_{21}(0) = 1, \quad G_{21}(\infty) := \lim_{x \uparrow \infty} G_{21}(x) = \frac{1}{2} \left( e^{-\int_0^\infty \Pi_2(dx)} + e^{\int_0^\infty \Pi_2(dx)} \right), \quad (2.18)$$

and

$$G_{22}(0) = 0, \quad G_{22}(\infty) := \lim_{x \uparrow \infty} G_{22}(x) = \frac{1}{2} \left( e^{\int_0^\infty \Pi_2(dx)} - e^{-\int_0^\infty \Pi_2(dx)} \right). \quad (2.19)$$

In particular,

$$G_{22}(\infty) < G_{21}(\infty) < e^{\int_0^\infty \Pi_2(dx)} \leq e^{\int_0^\infty \frac{1}{t} e^{-qt} (e^{-pt} - 1) dt}. \quad (2.20)$$

Corollary 2.1 states a similar result to Theorem 2.1, and can be proved by applying Theorem 2.1 to the dual process  $-X$ .

**Corollary 2.1** (i) For  $p, q > 0$ , there exists an infinitely divisible distribution  $G_3(x)$  on  $[0, \infty)$ , whose Laplace transform is given by

$$\int_{0-}^{\infty} e^{-sx} dG_3(x) = \frac{\mathbb{E} \left[ e^{sX_{e(q)}} \right]}{\mathbb{E} \left[ e^{sX_{e(p+q)}} \right]} = e^{\int_0^{\infty} (e^{-sx} - 1) \Pi_3(dx)}, \quad s \geq 0, \quad (2.21)$$

where

$$\Pi_3(dx) = \int_0^{\infty} \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(-X_t \in dx) dt, \quad x > 0. \quad (2.22)$$

(ii) For  $q > 0$  and  $-q < p < 0$ , there are two measures  $G_{41}(x)$  and  $G_{42}(x)$  on  $[0, \infty)$  such that

$$\int_{0-}^{\infty} e^{-sx} dG_{41}(x) = \frac{1}{2} \left( e^{-\int_0^{\infty} e^{-sx} \Pi_4(dx)} + e^{\int_0^{\infty} e^{-sx} \Pi_4(dx)} \right), \quad s > 0, \quad (2.23)$$

and

$$\int_{0-}^{\infty} e^{-sx} dG_{42}(x) = \frac{1}{2} \left( e^{\int_0^{\infty} e^{-sx} \Pi_4(dx)} - e^{-\int_0^{\infty} e^{-sx} \Pi_4(dx)} \right), \quad s > 0, \quad (2.24)$$

where  $\Pi_4(dx)$  is a measure and is given by

$$\Pi_4(dx) = \int_0^{\infty} \frac{1}{t} e^{-qt} (e^{-pt} - 1) \mathbb{P}(-X_t \in dx) dt, \quad x > 0. \quad (2.25)$$

(iii)  $G_3(x)$ ,  $G_{41}(x)$  and  $G_{42}(x)$  are continuous on  $(0, \infty)$ .

*Proof* For  $t \geq 0$ , let  $X_t^1 = -X_t$ . If  $p, q > 0$ , Theorem 2.1 (i) leads to that there is an infinitely divisible distribution  $G_3(x)$  on  $[0, \infty)$  such that

$$\int_{0-}^{\infty} e^{-sx} dG_3(x) = \frac{\mathbb{E} \left[ e^{-s\overline{X^1}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s\overline{X^1}_{e(p+q)}} \right]} = e^{\int_0^{\infty} (e^{-sx} - 1) \Pi_3(dx)}, \quad s \geq 0, \quad (2.26)$$

where

$$\Pi_3(dx) = \int_0^{\infty} \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t^1 \in dx) dt. \quad (2.27)$$

Then, formulas (2.21) and (2.22) are followed after replacing  $X^1$  by  $-X$  in (2.26) and (2.27). The proofs of (ii) and (iii) are similar, thus we omit the details.  $\square$

### 3 Main results

In this section, we first give a primary result (Theorem 3.1) in subsection 3.1, and then present some corollaries in subsection 3.2.

### 3.1 A primary result

For given  $y \geq b$ ,  $q > 0$  and  $p > -q$ , define

$$V_q(x) := \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \right], \quad x \in \mathbb{R}. \quad (3.1)$$

The following Theorem 3.1 is the primary result of this paper, whose proof is very long and is postponed to the later two sections.

**Theorem 3.1** *For  $q > 0$  and  $p > -q$ , we have*

$$V_q(x) - \mathbb{P}_x(X_{e(q)} > y) = J_1(b-x; y-b), \quad y \geq b, \quad (3.2)$$

with

$$J_1(x; y-b) = \int_{-\infty}^x F_1(x-z+y-b) dK_q(z), \quad x \in \mathbb{R}, \quad (3.3)$$

where  $K_q(x)$  is the convolution of the probability distribution functions of  $\underline{X}_{e(q)}$  and  $\overline{X}_{e(p+q)}$  under  $\mathbb{P}$ , i.e.,

$$K_q(x) = \int_{-\infty}^{\min\{0, x\}^+} \mathbb{P}(\overline{X}_{e(p+q)} \leq x-z) \mathbb{P}(\underline{X}_{e(q)} \in dz), \quad x \in \mathbb{R}, \quad (3.4)$$

and  $F_1(x)$  is a continuous function on  $(0, \infty)$  with the Laplace transform

$$\int_0^\infty e^{-sx} F_1(x) dx = \frac{1}{s} \left( \frac{\mathbb{E}[e^{-s\overline{X}_{e(q)}}]}{\mathbb{E}[e^{-s\overline{X}_{e(p+q)}}]} - 1 \right), \quad s > 0. \quad (3.5)$$

Before going further, we give some properties of  $K_q(x)$  in (3.4) and  $F_1(x)$  in (3.5) in Propositions 3.1 and 3.2, respectively. These two propositions are important for the rest of the paper.

**Proposition 3.1**  $K_q(x)$  in (3.4) is continuous on  $(-\infty, \infty)$ .

*Proof* If 0 is regular for  $(0, \infty)$  or  $(-\infty, 0)$ , i.e.,  $\mathbb{P}(\tau^+ = 0) = 1$  or  $\mathbb{P}(\tau^- = 0) = 1$ , where  $\tau^+ = \inf\{t > 0, X_t > 0\}$  and  $\tau^- = \inf\{t > 0, X_t < 0\}$ , then  $\mathbb{P}(\overline{X}_{e(q)} = z) = 0$  or  $\mathbb{P}(\underline{X}_{e(q)} = z) = 0$  for any  $q > 0$  and all  $z \in \mathbb{R}^1$ , thus the result that  $K_q(x)$  is continuous on  $\mathbb{R}$  is followed.

From Theorem 6.5 on page 142 and Corollary 6.6 on page 144 in Kyprianou [10], we obtain that 0 is irregular for both  $(0, \infty)$  and  $(-\infty, 0)$  (i.e.,  $\mathbb{P}(\tau^+ = 0) = \mathbb{P}(\tau^- = 0) = 0$ ) only when  $X$  is a compound Poisson process. Since the compound Poisson process is excluded in this paper, it must hold that 0 is regular for  $(0, \infty)$  or  $(-\infty, 0)$ . Thus the desired result is derived.  $\square$

<sup>1</sup> For some  $z \in \mathbb{R}$  and  $T > 0$ , if  $\int_0^T \mathbf{1}_{\{\overline{X}_t = z\}} dt > 0$ , then there is at least one interval  $(a, b)$  such that  $\overline{X}_t = z$  for all  $t \in (a, b)$  as the paths of  $\overline{X}$  are non-decreasing. Since 0 is regular for  $(0, \infty)$ , the probability  $\mathbb{P}(\overline{X}_t = z \text{ for all } t \in (a, b))$  is zero, where  $a < b$ . This gives  $\mathbb{E} \left[ \int_0^T \mathbf{1}_{\{\overline{X}_t = z\}} dt \right] = 0$  for all  $T > 0$  and  $z \in \mathbb{R}$ , thus  $\mathbb{P}(\overline{X}_{e(q)} = z) = 0$  for all  $q > 0$ .

**Proposition 3.2** (i) For  $q > 0$  and  $p > -q$ , it holds that

$$F_1(x) = \begin{cases} G_1(x) - 1, & \text{if } p > 0, \\ e^{\int_0^\infty \Pi_2(dx)} (G_{21}(x) - G_{22}(x)) - 1, & \text{if } p < 0, \end{cases} \quad (3.6)$$

where  $G_1(x)$ ,  $\Pi_2(dx)$ ,  $G_{21}(x)$  and  $G_{22}(x)$  are given by Theorem 2.1. Moreover,

$$F_1(0) := \lim_{x \downarrow 0} F_1(x) = e^{-\int_0^\infty \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t > 0) dt} - 1, \quad F_1(\infty) := \lim_{x \uparrow \infty} F_1(x) = 0. \quad (3.7)$$

(ii) For  $p > 0$ ,  $F_1(x)$  is continuous, increasing and bounded on  $[0, \infty]$ , and

$$-1 \leq F_1(x) \leq 0, \quad \text{for any } x \geq 0. \quad (3.8)$$

(iii) For  $-q < p < 0$  and  $x \geq 0$ ,

$$|F_1(x)| < 2e^{2 \int_0^\infty \frac{1}{t} e^{-qt} (e^{-pt} - 1) dt} + 1. \quad (3.9)$$

*Proof* (i) Applying integration by parts to (2.3) leads to

$$\int_0^\infty e^{-sx} G_1(x) dx = \frac{1}{s} \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+q)}} \right]}, \quad (3.10)$$

which combined with (3.5), yields

$$\int_0^\infty e^{-sx} F_1(x) dx = \int_0^\infty e^{-sx} (G_1(x) - 1) dx, \quad s > 0. \quad (3.11)$$

This gives  $F_1(x) = G_1(x) - 1$ , thus (3.6) holds for  $p > 0$ . Similarly, from (2.8), we can show that (3.6) is also valid for  $p < 0$ .

Then, noting that

$$F_1(0) = \lim_{s \uparrow \infty} \int_0^\infty s e^{-sx} F_1(x) dx \quad \text{and} \quad F_1(\infty) = \lim_{s \uparrow 0} \int_0^\infty s e^{-sx} F_1(x) dx, \quad (3.12)$$

we can derive (3.7) from (2.3), (2.8) and (3.5).

(ii) This result can be obtained from (3.6) since  $G_1(x)$  is a probability distribution function and is continuous (see Theorem 2.1 (i)).

(iii) This result is due to (2.20) and (3.6) since

$$|F_1(x)| < e^{\int_0^\infty \Pi_2(dx)} 2G_{21}(\infty) + 1.$$

□

**Remark 3.1** Since  $G_1(x)$ ,  $G_{21}(x)$  and  $G_{22}(x)$  are measures. Formula (3.6) means that  $F_1(x)$  can be written as

$$F_1(x) - F_1(0) = \int_0^x F_1(dz), \quad x > 0,$$

where for  $z > 0$ ,

$$F_1(dz) = \begin{cases} G_1(dz), & \text{if } p > 0, \\ e^{\int_0^\infty \Pi_2(dx)} (G_{21}(dz) - G_{22}(dz)), & \text{if } p < 0. \end{cases}$$



**Remark 3.2** Since  $F_1(dx)$  for  $x > 0$  is well defined, for given  $q > 0$  and  $p > -q$ , formula (3.5) gives

$$\int_0^\infty e^{-sx} F_1(dx) + F_1(0) + 1 = \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+q)}} \right]}, \quad s > 0. \quad (3.13)$$

As  $F_1(x)$  is bounded and continuous on  $[0, \infty]$  (see Proposition 3.2), the above identity (3.13) can be extended to the half-plane  $\text{Re}(s) \geq 0$ . Particularly,

$$\int_{0^-}^\infty e^{i\phi x} d(F_1(x) + 1) = \frac{\mathbb{E} \left[ e^{i\phi \bar{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{i\phi \bar{X}_{e(p+q)}} \right]}, \quad \text{for } \phi \in \mathbb{R}. \quad (3.14)$$

**Remark 3.3** If  $y = b$  in (3.1), then for fixed  $b \in \mathbb{R}$ , it follows from (3.2)–(3.5) and (3.14) that

$$\int_{-\infty}^\infty e^{-i\phi(x-b)} dV_q(x) = \mathbb{E} \left[ e^{i\phi \underline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{i\phi \bar{X}_{e(p+q)}} \right], \quad \phi \in \mathbb{R}. \quad (3.15)$$

In particular, if  $p = 0$ , then (3.15) will reduce to the following well-known Wiener-Hopf factorization (see, e.g., Theorem 6.16 in [10])

$$\mathbb{E} \left[ e^{i\phi \underline{X}_{e(q)}} \right] = \mathbb{E} \left[ e^{i\phi \underline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{i\phi \bar{X}_{e(q)}} \right], \quad \phi \in \mathbb{R}.$$

### 3.2 Some corollaries

**Corollary 3.1** For  $q > 0$  and  $p > -q$ ,

$$\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \geq b\}} ds} \mathbf{1}_{\{X_{e(q)} < y\}} \right] - \mathbb{P}_x (X_{e(q)} < y) = J_2(x - b; b - y), \quad y \leq b, \quad (3.16)$$

with

$$J_2(x; b - y) = \int_{-\infty}^x F_2(x - z + b - y) dL_q(z), \quad x \in \mathbb{R}, \quad (3.17)$$

where  $L_q(x)$  is the convolution of the probability distribution functions of  $-\underline{X}_{e(p+q)}$  and  $-\bar{X}_{e(q)}$  under  $\mathbb{P}$ , i.e.,

$$L_q(x) = \int_{\max\{0, x\}^-}^\infty \mathbb{P}(-\bar{X}_{e(q)} \leq x - z) \mathbb{P}(-\underline{X}_{e(p+q)} \in dz), \quad x \in \mathbb{R}, \quad (3.18)$$

and  $F_2(x)$  is a continuous function on  $(0, \infty)$  with the Laplace transform

$$\int_0^\infty e^{-sz} F_2(z) dz = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{s \underline{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{s \underline{X}_{e(p+q)}} \right]} - 1 \right), \quad s > 0. \quad (3.19)$$

*Proof* Consider the dual process  $X_t^1 = -X_t$  for  $t \geq 0$ . For  $q > 0$  and  $p > -q$ , the convolution of the probability distribution functions of  $\underline{X}_{e(q)}^1$  and  $\overline{X}_{e(p+q)}^1$  under  $\mathbb{P}$  is given by  $L_q(x)$  in (3.18).

As  $y \leq b$ , i.e.,  $-y \geq -b$ , we can obtain from Theorem 3.1 that

$$\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s^1 \leq -b\}} ds} \mathbf{1}_{\{X_{e(q)}^1 > -y\}} \right] - \mathbb{P}_x \left( X_{e(q)}^1 > -y \right) = J_2(-b-x; b-y), \quad (3.20)$$

with

$$J_2(x; b-y) = \int_{-\infty}^x F_2(x-z-y+b) dL_q(z), \quad x \in \mathbb{R}, \quad (3.21)$$

where  $F_2(x)$  is given by (3.19) since

$$\int_0^\infty e^{-sx} F_2(x) dx = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{-s \overline{X}_{e(q)}^1} \right]}{\mathbb{E} \left[ e^{-s \overline{X}_{e(p+q)}^1} \right]} - 1 \right) = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{s \underline{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{s \underline{X}_{e(p+q)}} \right]} - 1 \right).$$

In addition, it is obvious that

$$\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s^1 \leq -b\}} ds} \mathbf{1}_{\{X_{e(q)}^1 > -y\}} \right] = \mathbb{E}_{-x} \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \geq b\}} ds} \mathbf{1}_{\{X_{e(q)} < y\}} \right], \quad (3.22)$$

and

$$\mathbb{P}_x \left( X_{e(q)}^1 > -y \right) = \mathbb{P}_{-x} \left( X_{e(q)} < y \right). \quad (3.23)$$

Thus, formula (3.16) is derived from (3.20) by first using the last two formulas and then replacing  $-x$  by  $x$ .  $\square$

**Remark 3.4** Similar to the derivation of (3.6), we can show that

$$F_2(x) = \begin{cases} G_3(x) - 1, & \text{if } p > 0, \\ e^{\int_0^\infty \Pi_4(dx)} \left( G_{41}(x) - G_{42}(x) \right) - 1, & \text{if } p < 0, \end{cases} \quad (3.24)$$

where  $G_3(x)$ ,  $\Pi_4(dx)$ ,  $G_{41}(x)$  and  $G_{42}(x)$  are given by Corollary 2.1.

**Remark 3.5** Similar to the derivation of (3.15), for fixed  $b \in \mathbb{R}$ , we can deduce the following result from (3.16).

$$\int_{-\infty}^\infty e^{-i\phi(x-b)} d \left( \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \geq b\}} ds} \mathbf{1}_{\{X_{e(q)} < b\}} \right] \right) = -\mathbb{E} \left[ e^{i\phi \underline{X}_{e(p+q)}} \right] \mathbb{E} \left[ e^{i\phi \overline{X}_{e(q)}} \right], \quad \phi \in \mathbb{R}.$$

**Corollary 3.2** (i) For  $p, q > 0$  and  $y \geq b$ , we have

$$\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s > b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \right] = \frac{q}{p+q} \left( \mathbb{P}_x \left( X_{e(p+q)} > y \right) + \hat{J}_1(b-x; y-b) \right),$$

where

$$\hat{J}_1(x; y-b) = \int_{-\infty}^x \hat{F}_1(x-z+y-b) d\hat{K}_q(z), \quad x \in \mathbb{R}. \quad (3.25)$$

Here, in (3.25),  $\hat{K}_q(x)$  is the convolution of the probability distribution functions of  $\underline{X}_{e(p+q)}$  and  $\bar{X}_{e(q)}$  under  $\mathbb{P}$ ;  $\hat{F}_1(x)$  is a continuous function on  $(0, \infty)$  and satisfies

$$\int_0^\infty e^{-sx} \hat{F}_1(x) dx = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+q)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]} - 1 \right), \quad s > 0. \quad (3.26)$$

(ii) For  $p, q > 0$  and  $y \leq b$ ,

$$\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s < b\}} ds} \mathbf{1}_{\{X_{e(q)} < y\}} \right] = \frac{q}{p+q} \left( \mathbb{P}_x(X_{e(p+q)} < y) + \hat{f}_2(x-b; b-y) \right),$$

where

$$\hat{f}_2(x; b-y) = \int_{-\infty}^x \hat{F}_2(x-z-y+b) d\hat{L}_q(z), \quad x \in \mathbb{R}. \quad (3.27)$$

In (3.27),  $\hat{L}_q(x)$  is the convolution of the probability distribution functions of  $-\underline{X}_{e(q)}$  and  $-\bar{X}_{e(p+q)}$  under  $\mathbb{P}$ ;  $\hat{F}_2(x)$  is a continuous function on  $(0, \infty)$  and satisfies

$$\int_0^\infty e^{-sx} \hat{F}_2(x) dx = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{s\underline{X}_{e(p+q)}} \right]}{\mathbb{E} \left[ e^{s\underline{X}_{e(q)}} \right]} - 1 \right), \quad s > 0. \quad (3.28)$$

*Proof* Note first that

$$\mathbb{E}_x \left[ e^{-p \int_0^t \mathbf{1}_{\{X_s > b\}} ds} \mathbf{1}_{\{X_t > y\}} \right] = e^{-pt} \mathbb{E}_x \left[ e^{-(-p) \int_0^t \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_t > y\}} \right]. \quad (3.29)$$

Then it holds that

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_{e(q)} > b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \right] \\ &= q \int_0^\infty e^{-qt} e^{-pt} \mathbb{E}_x \left[ e^{p \int_0^t \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_t > y\}} \right] dt \\ &= \frac{q}{p+q} \mathbb{E}_x \left[ e^{-(-p) \int_0^{e(p+q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(p+q)} > y\}} \right], \end{aligned} \quad (3.30)$$

which combined with (3.2) and (3.5), gives the results in the first part. The derivation of the second part is similar and thus we omit the details.  $\square$

**Remark 3.6** Since Lemma 2.1 holds, the item  $\int_0^{e(q)} \mathbf{1}_{\{X_s < b\}} ds$  in (3.27) can be rewritten as  $\int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds$ . A similar result holds for the quantity  $\int_0^{e(q)} \mathbf{1}_{\{X_s > b\}} ds$  in (3.25).

**Remark 3.7** Although it is assumed that  $p, q > 0$  in Corollary 3.2, one can verify that this corollary also holds for  $q > 0$  and  $p > -q$  by using a similar derivation in Theorem 3.1. The reason why we focus on the case  $p, q > 0$  in Corollary 3.2 is that now it is a straightforward result of Theorem 3.1 and Corollary 3.1.

Similar to Remark 3.1, for the three functions  $F_2(x)$  in (3.19),  $\hat{F}_1(x)$  in (3.26) and  $\hat{F}_2(x)$  in (3.28), we have expressions for  $F_2(dx)$ ,  $\hat{F}_1(dx)$  and  $\hat{F}_2(dx)$  for  $x > 0$ . Particularly, Theorem 3.1 and Corollaries 3.1 and 3.2 will give us the following result.

**Corollary 3.3** (i) For  $p, q > 0$ , we have

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] \\ &= \begin{cases} \mathbb{P}_x(X_{e(q)} \in dy) - \int_{-\infty}^{b-x} F_1(dy-x-z) dK_q(z), & y \geq b, \\ \frac{q}{\xi} \left( \mathbb{P}_x(X_{e(\xi)} \in dy) - \int_{-\infty}^{x-b} \hat{F}_2(x-dy-z) d\hat{L}_q(z) \right), & y \leq b, \end{cases} \end{aligned} \quad (3.31)$$

where  $\xi = p + q$ .

(ii) For  $p, q > 0$ , we have

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \geq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] \\ &= \begin{cases} \mathbb{P}_x(X_{e(q)} \in dy) - \int_{-\infty}^{x-b} F_2(x-dy-z) dL_q(z), & y \leq b, \\ \frac{q}{\xi} \left( \mathbb{P}_x(X_{e(\xi)} \in dy) - \int_{-\infty}^{b-x} \hat{F}_1(dy-x-z) d\hat{K}_q(z) \right), & y \geq b. \end{cases} \end{aligned} \quad (3.32)$$

**Remark 3.8** To obtain closed-form formulas for  $\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right]$  in (3.31), it is enough to know the distributions of  $\overline{X}_{e(q)}$  and  $\underline{X}_{e(q)}$  for any  $q > 0$ , because these two distributions determine the distribution of  $X_{e(q)}$  (which is due to the Wiener-Hopf factorization; see, e.g., Theorem 6.16 in [10]) and other quantities, i.e.,  $F_1(x)$ ,  $\hat{F}_2(x)$ ,  $K_q(x)$  and  $\hat{L}_q(x)$ . The distributions of  $\overline{X}_{e(q)}$  and  $\underline{X}_{e(q)}$  have been investigated considerably, the reader can refer to [7,9].

**Remark 3.9** It follows from (3.14) that

$$\int_0^\infty e^{i\phi x} F_1(dx) + F_1(0) + 1 = \mathbb{E} \left[ e^{i\phi \overline{X}_{e(q)}} \right] / \mathbb{E} \left[ e^{i\phi \overline{X}_{e(p+q)}} \right], \text{ for } \phi \in \mathbb{R}.$$

which combined with the definition of  $K_q(x)$  in (3.4), gives

$$\begin{aligned} & \int_{-\infty}^\infty e^{i\phi x} \int_{-\infty}^x F_1(dx-z) dK_q(z) = \int_0^\infty e^{i\phi x} F_1(dx) \int_{-\infty}^\infty e^{i\phi x} dK_q(x) \\ &= \mathbb{E} \left[ e^{i\phi X_{e(q)}} \right] - (F_1(0) + 1) \int_{-\infty}^\infty e^{i\phi x} dK_q(x), \quad \phi \in \mathbb{R}, \end{aligned}$$

where in the second equality, we have used the known Wiener-Hopf factorization (see Remark 3.3). Therefore, for  $x \in \mathbb{R}$ , the last formula produces

$$\int_{-\infty}^x F_1(dx-z) dK_q(z) = \mathbb{P}(X_{e(q)} \in dx) - (F_1(0) + 1) K_q(dx).$$

Similarly, we can derive that

$$\begin{aligned} & \int_{-\infty}^x \hat{F}_2(dx-z) d\hat{L}_q(z) = \mathbb{P}(X_{e(p+q)} \in -dx) - (\hat{F}_2(0) + 1) \hat{L}_q(dx), \\ & \int_{-\infty}^x \hat{F}_1(dx-z) d\hat{K}_q(z) = \mathbb{P}(X_{e(p+q)} \in dx) - (\hat{F}_1(0) + 1) \hat{K}_q(dx), \end{aligned}$$

and

$$\int_{-\infty}^x F_2(dx-z) dL_q(z) = \mathbb{P}(X_{e(q)} \in -dx) - (F_2(0) + 1) L_q(dx).$$

**Remark 3.10** From the above remark, we can write (3.31) and (3.32) as

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] \\ &= \begin{cases} (F_1(0) + 1)K_q(dy - x) + \int_{b-x}^{y-x} F_1(dy - x - z) dK_q(z), & y \geq b, \\ \frac{q}{p+q} \left( (\hat{F}_2(0) + 1)\hat{L}_q(x - dy) + \int_{x-b}^{x-y} \hat{F}_2(x - dy - z) d\hat{L}_q(z) \right), & y \leq b, \end{cases} \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \geq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] \\ &= \begin{cases} (F_2(0) + 1)L_q(x - dy) + \int_{x-b}^{x-y} F_2(x - dy - z) dL_q(z), & y \leq b, \\ \frac{q}{p+q} \left( (\hat{F}_1(0) + 1)\hat{K}_q(dy - x) + \int_{b-x}^{y-x} \hat{F}_1(dy - x - z) d\hat{K}_q(z) \right), & y \geq b, \end{cases} \end{aligned} \quad (3.34)$$

where  $q, p > 0$ .

#### 4 Proof of Theorem 3.1 in a dense subclass

In this section, the process  $X = (X_t)_{t \geq 0}$  is assumed to be a Lévy process with Gaussian component and its jumps have rational Laplace transform, and one can refer to, e.g., Lewis and Mordecki [14] for the investigation on such processes. In specific, the process  $X$  is given by

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N_t^+} Z_k^+ - \sum_{k=1}^{N_t^-} Z_k^-, \quad (4.1)$$

where  $\mu$  and  $X_0$  are constants;  $(W_t)_{t \geq 0}$  is a standard Brownian motion with  $W_0 = 0$ , and  $\sigma > 0$  is the volatility of the diffusion;  $(N_t^+)_{t \geq 0}$  is a Poisson process with rate  $\lambda^+$ , and  $(N_t^-)_{t \geq 0}$  is a Poisson process with rate  $\lambda^-$ ;  $Z_k^+$  ( $Z_k^-$ ),  $k = 1, 2, \dots$ , are independent and identically distributed random variables; moreover,  $(W_t)_{t \geq 0}$ ,  $(N_t^+)_{t \geq 0}$ ,  $(N_t^-)_{t \geq 0}$ ,  $\{Z_k^+; k = 1, 2, \dots\}$  and  $\{Z_k^-; k = 1, 2, \dots\}$  are independent mutually; finally, the density functions of  $Z_1^+$  and  $Z_1^-$  are given respectively by

$$p^+(z) = \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} c_{kj} \frac{(\eta_k)^j z^{j-1}}{(j-1)!} e^{-\eta_k z}, \quad z > 0, \quad (4.2)$$

and

$$p^-(z) = \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} d_{kj} \frac{(\vartheta_k)^j z^{j-1}}{(j-1)!} e^{-\vartheta_k z}, \quad z > 0. \quad (4.3)$$

Besides, it is assumed that  $\eta_i \neq \eta_j$  and  $\vartheta_i \neq \vartheta_j$  for  $i \neq j$ .

The following Lemma 4.1 is a combination of Lemma 1.1 in [14] and Proposition 1 (v) in [8]. It characterizes the roots of  $\psi(z) = q$  with

$$\begin{aligned} \psi(z) := \ln(\mathbb{E}[e^{izX_1}]) &= \lambda^+ \left( \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} c_{kj} \left( \frac{\eta_k}{\eta_k - iz} \right)^j - 1 \right) \\ &+ iz\mu - \frac{\sigma^2}{2} z^2 + \lambda^- \left( \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} d_{kj} \left( \frac{\vartheta_k}{\vartheta_k + iz} \right)^j - 1 \right), \quad z \in \mathbb{R}, \end{aligned} \quad (4.4)$$

where  $\sigma > 0$ .

**Lemma 4.1** (i) For almost all  $q > 0$ , the equation  $\psi(z) = q$  has, in the set  $\text{Im}(z) < 0$ , a total of  $M = \sum_{k=1}^{m^+} m_k + 1$  distinct simple solutions  $-i\beta_{1,q}, -i\beta_{2,q}, \dots, -i\beta_{M,q}$ , ordered such that

$$0 < \beta_{1,q} < \text{Re}(\beta_{2,q}) \leq \dots \leq \text{Re}(\beta_{M,q}). \quad (4.5)$$

(ii) For almost all  $q > 0$ , the equation  $\psi(z) = q$  has, in the set  $\text{Im}(z) > 0$ , a total of  $N = \sum_{k=1}^{n^-} n_k + 1$  distinct simple roots  $i\gamma_{1,q}, i\gamma_{2,q}, \dots, i\gamma_{N,q}$ , ordered such that

$$0 < \gamma_{1,q} < \text{Re}(\gamma_{2,q}) \leq \dots \leq \text{Re}(\gamma_{N,q}). \quad (4.6)$$

(iii) There are only finite numbers of  $q > 0$  such that  $\psi(z) = q$  has a root with multiplicity larger than one.

From now on, we denote by  $\mathbb{Q}$  the set of  $q > 0$  such that the equation  $\psi(z) = q$  only has simple roots.

Our objection in this section is proving that Theorem 3.1 holds for the process  $X$  in (4.1), and this will be done in Subsections 4.1 and 4.2.

#### 4.1 The case of $q, p + q \in \mathbb{Q}$

In this subsection, we want to show that Theorem 3.1 holds for  $X$  given by (4.1) and  $q, p + q \in \mathbb{Q}$ . First, for  $y > b$  and  $q, p + q \in \mathbb{Q}$ , the expression for  $V_q(x) = \mathbb{E}_x \left[ e^{-p \int_0^{\epsilon(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{\epsilon(q)} > y\}} \right]$  with  $X$  in (4.1) is summarized in Proposition 4.1, whose proof is left to the Appendix.

**Proposition 4.1** For  $X$  in (4.1),  $y > b$ , and  $q, p + q \in \mathbb{Q}$ ,

$$V_q(x) = \begin{cases} \sum_{k=1}^M U_k e^{\beta_{k,p+q}(x-b)}, & x < b, \\ \sum_{k=1}^M H_k e^{\beta_{k,q}(x-y)} + \sum_{k=1}^N P_k e^{\gamma_{k,q}(b-x)}, & b < x < y, \\ 1 + \sum_{k=1}^N Q_k e^{\gamma_{k,q}(y-x)} + \sum_{k=1}^N P_k e^{\gamma_{k,q}(b-x)}, & x > y, \end{cases} \quad (4.7)$$

where  $H_k$  and  $Q_k$  are given by (A.18) and (A.19);  $U_k$  and  $P_k$  are given by rational expansion:

$$\begin{aligned} & \sum_{i=1}^M \frac{U_i}{x - \beta_{i,p+q}} - \sum_{i=1}^N \frac{P_i}{x + \gamma_{i,q}} - \sum_{i=1}^M \frac{H_i}{x - \beta_{i,q}} e^{\beta_{i,q}(b-y)} \\ &= \frac{\prod_{k=1}^{m^+} (x - \eta_k)^{m_k} \prod_{k=1}^{n^-} (x + \vartheta_k)^{n_k}}{\prod_{i=1}^M (x - \beta_{i,p+q}) \prod_{i=1}^N (x + \gamma_{i,q})} \times \\ & \sum_{k=1}^M \frac{\prod_{i=1}^M (\beta_{k,q} - \beta_{i,p+q}) \prod_{i=1}^N (\beta_{k,q} + \gamma_{i,q})}{\prod_{i=1}^{m^+} (\beta_{k,q} - \eta_i)^{m_i} \prod_{i=1}^{n^-} (\beta_{k,q} + \vartheta_i)^{n_i}} \frac{-H_k}{x - \beta_{k,q}} e^{\beta_{k,q}(b-y)}. \end{aligned} \quad (4.8)$$

**Remark 4.1** The expressions for  $U_k$  and  $P_k$  can be easily obtained from (4.8), but here, we are not interested in these expressions. Hence, the corresponding results are omitted for the sake of brevity.

**Remark 4.2** We comment that  $V_q(x)$  in (4.7) is continuous on  $(-\infty, \infty)$ . In fact, from (4.7), it is enough to show

$$V_q(b-) = V_q(b+) \text{ and } V_q(y-) = V_q(y+). \quad (4.9)$$

These two identities can be derived from (A.22) and the following result:

$$\sum_{i=1}^M H_i - \sum_{i=1}^N Q_i - 1 = 0, \quad (4.10)$$

which can be obtained from (A.24) by letting  $\theta \uparrow \infty$ .

**Lemma 4.2** For  $X$  in (4.1),  $y > b$ , and  $q, p+q \in \mathbb{Q}$ , we have

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\phi(x-b)} (V_q(x) - \mathbb{P}_x(X_{e(q)} > y)) dx \\ &= \mathbb{E} \left[ e^{\phi \underline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{\phi \overline{X}_{e(p+q)}} \right] \int_0^{\infty} F_0(x+y-b) e^{\phi x} dx, \quad \operatorname{Re}(\phi) = 0, \end{aligned} \quad (4.11)$$

where

$$F_0(x) = \sum_{i=1}^M e^{-\beta_{i,q}x} \prod_{k=1}^M \frac{\beta_{i,q} - \beta_{k,p+q}}{\beta_{k,p+q}} \prod_{k=1, k \neq i}^M \frac{\beta_{k,q}}{\beta_{i,q} - \beta_{k,q}}, \quad x \geq 0. \quad (4.12)$$

In addition,

$$\begin{aligned} \int_0^{\infty} e^{-sx} F_0(x) dx &= \sum_{i=1}^M \prod_{k=1}^M \frac{\beta_{i,q} - \beta_{k,p+q}}{\beta_{k,p+q}} \prod_{k=1, k \neq i}^M \frac{\beta_{k,q}}{\beta_{i,q} - \beta_{k,q}} \frac{1}{\beta_{i,q} + s} \\ &= \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{-s \overline{X}_{e(q)}} \right]}{\mathbb{E} \left[ e^{-s \overline{X}_{e(p+q)}} \right]} - 1 \right), \quad s > 0. \end{aligned} \quad (4.13)$$

*Proof* From (4.7), for  $Re(\phi) = 0$ , some direct calculations yield

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\phi(x-b)} dV_q(x) &= \sum_{i=1}^M \frac{U_i \phi}{\beta_{i,p+q} - \phi} + \sum_{i=1}^N \frac{P_i \phi}{\phi + \gamma_{i,q}} \\
&\quad - \sum_{i=1}^M \frac{H_i \phi}{\beta_{i,q} - \phi} e^{\beta_{i,q}(b-y)} + e^{\phi(b-y)} \left( \sum_{i=1}^M \frac{H_i \beta_{i,q}}{\beta_{i,q} - \phi} - \sum_{i=1}^N \frac{Q_i \gamma_{i,q}}{\gamma_{i,q} + \phi} \right) \\
&= \phi \mathbb{E} \left[ e^{\phi \underline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{\phi \overline{X}_{e(p+q)}} \right] \int_0^{\infty} F_0(x+y-b) e^{\phi x} dx \\
&\quad + e^{\phi(b-y)} \left( \sum_{i=1}^M \frac{H_i \beta_{i,q}}{\beta_{i,q} - \phi} - \sum_{i=1}^N \frac{Q_i \gamma_{i,q}}{\gamma_{i,q} + \phi} \right) \\
&= \phi \mathbb{E} \left[ e^{\phi \underline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{\phi \overline{X}_{e(p+q)}} \right] \int_0^{\infty} F_0(x+y-b) e^{\phi x} dx + e^{\phi(b-y)} \psi_q^+(-\phi) \psi_q^-(\phi),
\end{aligned} \tag{4.14}$$

where  $\psi_q^+(\cdot)$  and  $\psi_q^-(\cdot)$  are given respectively by (A.1) and (A.4); in the first equality, we have used the identity  $\sum_{i=1}^M (U_i - H_i e^{\beta_{i,q}(b-y)}) - \sum_{i=1}^N P_i = 0$  (see (A.22)); the second equality follows from (4.8), (A.1), (A.4) and (A.18) with  $F_0(x)$  given by (4.12); the third one is due to (4.10) and (A.24).

For fixed  $y$ , we have

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\phi(x-y)} d\mathbb{P}_x(X_{e(q)} > y) &= \mathbb{E} \left[ e^{\phi X_{e(q)}} \right] \\
&= \mathbb{E} \left[ e^{\phi \overline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{\phi \underline{X}_{e(q)}} \right] = \psi_q^+(-\phi) \psi_q^-(\phi),
\end{aligned} \tag{4.15}$$

where the second equality is due to the well-known Wiener-Hopf factorization (see, e.g., Theorem 6.16 in [10]). Then, from (4.15), applying integration by parts will lead to

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\phi(x-b)} dV_q(x) - e^{\phi(b-y)} \psi_q^+(-\phi) \psi_q^-(\phi) \\
= \phi \int_{-\infty}^{\infty} e^{-\phi(x-b)} (V_q(x) - \mathbb{P}_x(X_{e(q)} > y)) dx.
\end{aligned} \tag{4.16}$$

From (4.14) and (4.16), we arrive at (4.11). The second equality in (4.13) is a direct result of (A.1) and (4.17) in the following Lemma 4.3.  $\square$

**Lemma 4.3** For given constants  $\tilde{x}_1, \dots, \tilde{x}_n$ , which satisfy  $\tilde{x}_i \neq \tilde{x}_j$  for  $i \neq j$ , it holds that

$$\sum_{i=1}^n \frac{\prod_{k=1}^m (\tilde{x}_i - \hat{x}_k)}{\prod_{k=1, k \neq i}^n (\tilde{x}_i - \tilde{x}_k)} = 0, \tag{4.17}$$

where  $m < n - 1$  and  $\hat{x}_1, \dots, \hat{x}_m$  are arbitrary constants.

*Proof* Note first that

$$\frac{x \prod_{i=1}^m (x - \hat{x}_i)}{\prod_{i=1}^n (x - \tilde{x}_i)} = \sum_{i=1}^n \frac{\prod_{k=1}^m (\tilde{x}_i - \hat{x}_k)}{\prod_{k=1, k \neq i}^n (\tilde{x}_i - \tilde{x}_k)} \frac{x}{x - \tilde{x}_i}. \tag{4.18}$$

Since  $m < n - 1$ , formula (4.17) is derived from (4.18) by letting  $x \uparrow \infty$ .  $\square$



**Proposition 4.2** For  $X$  in (4.1),  $q > 0$  and  $p > -q$  such that  $q, p + q \in \mathbb{Q}$ , Theorem 3.1 holds, i.e.,

$$V_q(x) - \mathbb{P}_x(X_{e(q)} > y) = J_0(b - x; y - b), \quad y \geq b, \quad (4.19)$$

with

$$J_0(x; y - b) = \int_{-\infty}^x F_0(x - z + y - b) dK_q(z), \quad x \in \mathbb{R}, \quad (4.20)$$

where  $K_q(x)$  is given by (3.4) with  $X$  in (4.1);  $F_0(x)$  is given by (4.12) and its Laplace transform is given by (4.13).

*Proof* Since  $V_q(x)$  is a continuous function of  $x$  (see Remark 4.2) and  $\mathbb{P}_x(X_{e(q)} > y)$ , as a function of  $x$ , is also continuous with respect to  $x$  (as  $\mathbb{P}(X_{e(q)} = z) = 0$  for all  $z \in \mathbb{R}$ , see Lemma 2.1), the integrand on the left-hand side of (4.11) is continuous with respect to  $x$ .

As the distributions of  $\bar{X}_{e(p+q)}$  and  $\underline{X}_{e(q)}$  have continuous density functions (see Lemma A.1),  $K_q(x)$  (see (3.4)) also has a density function. This result and the continuity of  $F_0(x)$  on  $(0, \infty)$  will lead to that  $J_0(x; y - b)$  in (4.20) is continuous on  $(-\infty, \infty)$ . Finally, the definition of  $J_0(x; y - b)$  gives

$$\mathbb{E} \left[ e^{\phi \underline{X}_{e(q)}} \right] \mathbb{E} \left[ e^{\phi \bar{X}_{e(p+q)}} \right] \int_0^\infty F_0(x + y - b) e^{\phi x} dx = \int_{-\infty}^\infty J_0(x; y - b) e^{\phi x} dx. \quad (4.21)$$

From (4.11) and (4.21), we can derive the conclusion that (4.19) holds for  $y > b$ .

Next, it is obvious that

$$\lim_{y \downarrow b} \mathbb{P}_x(X_{e(q)} > y) = \mathbb{P}_x(X_{e(q)} > b). \quad (4.22)$$

Besides, we have  $\lim_{y \downarrow b} J_0(x; y - b) = J_0(x; 0)$ , which is due to the fact that  $F_0(x)$  is bounded and continuous on  $(0, \infty)$  (see (4.12)) and the dominated convergence theorem. Since (A.14) holds, applying the dominated convergence theorem again gives

$$\lim_{y \downarrow b} \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \right] = \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > b\}} \right].$$

Therefore, we conclude that (4.19) also holds for  $y = b$ .  $\square$

#### 4.2 The case of $q \in \mathbb{Q}^c$ or $p + q \in \mathbb{Q}^c$

In this subsection, for given  $q > 0$  and  $p > -q$ , we assume that either  $q \in \mathbb{Q}^c$  or  $p + q \in \mathbb{Q}^c$  holds.

From Lemma 4.1 (iii), we can find a sequence of  $q_n$  such that  $q_n, p + q_n \in \mathbb{Q}$  and  $\lim_{n \uparrow \infty} q_n \downarrow q$ . For each  $n$ , the result in Proposition 4.2 leads to

$$V_{q_n}(x) - \mathbb{P}_x(X_{e(q_n)} > y) = J_0^n(b - x; y - b), \quad y > b, \quad (4.23)$$

with

$$J_0^n(x; y - b) = \int_{-\infty}^x F_0^n(x - z + y - b) dK_{q_n}(z), \quad x \in \mathbb{R}, \quad (4.24)$$

where  $K_{q_n}(x)$  is the convolution of  $\underline{X}_{e(q_n)}$  and  $\overline{X}_{e(p+q_n)}$  under  $\mathbb{P}$ , and

$$\int_0^\infty e^{-sx} F_0^n(x) dx = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{-s\overline{X}_{e(q_n)}} \right]}{\mathbb{E} \left[ e^{-s\overline{X}_{e(p+q_n)}} \right]} - 1 \right), \quad s > 0. \quad (4.25)$$

**Lemma 4.4** *It holds that*

$$\lim_{n \uparrow \infty} V_{q_n}(x) = V_q(x), \quad \lim_{n \uparrow \infty} \mathbb{P}_x(X_{e(q_n)} > y) = \mathbb{P}_x(X_{e(q)} > y). \quad (4.26)$$

And for  $K_q(x)$  given by (3.4) with  $X$  in (4.1), we have

$$\lim_{n \uparrow \infty} K_{q_n}(x) = K_q(x), \quad x \in \mathbb{R}. \quad (4.27)$$

*Proof* It follows from the definition of (3.1) that

$$V_{q_n}(x) = \int_0^\infty q_n e^{-q_n t} \mathbb{E}_x \left[ e^{-p \int_0^t \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_t > y\}} \right] dt. \quad (4.28)$$

Since  $q_n > q$  and  $p + q > 0$ , formula (4.26) is obtained from the application of the dominated convergence theorem. Similarly, for  $\text{Re}(\phi) \geq 0$ , we have

$$\lim_{n \uparrow \infty} \mathbb{E} \left[ e^{\phi \underline{X}_{e(q_n)}} \right] = \mathbb{E} \left[ e^{\phi \underline{X}_{e(q)}} \right], \quad \lim_{n \uparrow \infty} \mathbb{E} \left[ e^{-\phi \overline{X}_{e(p+q_n)}} \right] = \mathbb{E} \left[ e^{-\phi \overline{X}_{e(p+q)}} \right]. \quad (4.29)$$

Formula (4.29) means that  $\overline{X}_{e(p+q_n)}$  and  $\underline{X}_{e(q_n)}$  converge respectively to  $\overline{X}_{e(p+q)}$  and  $\underline{X}_{e(q)}$  in distribution. Thus, recalling the definition of  $K_q(x)$  in (3.4) and Proposition 3.1, we derive (4.27).  $\square$

For the function  $F_0^n(x)$  in (4.25) and  $F_1(x)$  in (3.5) with  $X$  given by (4.1), we have the following lemma.

**Lemma 4.5** (i)  $F_0^n(x)$  is uniformly convergence to  $F_1(x)$  on  $[0, \infty]$ .  
(ii)  $F_1(x)$ ,  $F_0^1(x)$ ,  $F_0^2(x)$ ,  $\dots$ , are uniformly bounded.

*Proof* First, formula (3.7) leads to

$$F_0^n(\infty) = 0 = F_1(\infty), \quad (4.30)$$

and  $F_0^n(0) = e^{-\int_0^\infty \frac{1}{\tau} e^{-q_n t} (1 - e^{-p t}) \mathbb{P}(X_t > 0) dt} - 1$ . Thus

$$\lim_{n \uparrow \infty} F_0^n(0) = F_1(0) = e^{-\int_0^\infty \frac{1}{\tau} e^{-q t} (1 - e^{-p t}) \mathbb{P}(X_t > 0) dt} - 1. \quad (4.31)$$

In addition, similar to (4.29), we also have

$$\lim_{n \uparrow \infty} \mathbb{E} \left[ e^{-s \overline{X}_{e(q_n)}} \right] = \mathbb{E} \left[ e^{-s \overline{X}_{e(q)}} \right], \quad s > 0. \quad (4.32)$$

(1) Assume  $p > 0$ . It follows from (3.13), (4.29) and (4.32) that

$$\lim_{n \uparrow \infty} \int_{0-}^\infty e^{-sx} d(F_0^n(x) + 1) = \int_{0-}^\infty e^{-sx} d(F_1(x) + 1). \quad (4.33)$$

Since  $F_1(x)$  is continuous on  $(0, \infty)$ , applying the continuity theorem for Laplace transforms (see, e.g., Theorem 2a on page 433 in Feller [6]) to (4.33) gives

$$\lim_{n \uparrow \infty} F_0^n(x) + 1 = F_1(x) + 1, \quad x > 0. \quad (4.34)$$

Since  $F_0^n(x)$  is increasing, continuous and bounded on  $(0, \infty)$  (see Proposition 3.2 (ii)). From (4.30), (4.31) and (4.34), we deduce the result that  $F_0^n(x)$  is uniformly convergence to  $F_1(x)$  on  $[0, \infty]$ . Besides, formula (3.8) produces

$$-1 \leq F_1(x), \quad F_0^n(x) \leq 0, \quad \text{for } n = 1, 2, \dots, \quad (4.35)$$

(2) Assume  $-q < p < 0$ . For each  $n = 1, 2, \dots$ , we obtain from (3.6) that

$$F_0^n(x) = e^{-\Pi_2^n(0, \infty)} (G_{21}^n(x) - G_{22}^n(x)) - 1, \quad (4.36)$$

where

$$\Pi_2^n(0, \infty) := \int_0^\infty \frac{1}{t} e^{-qnt} (e^{-pt} - 1) \mathbb{P}(X_t > 0) dt, \quad (4.37)$$

and  $G_{21}^n(x)$  and  $G_{22}^n(x)$  are given respectively by (see (2.5), (2.6) and (2.8))

$$2 \int_{0^-}^\infty e^{-sx} dG_{21}^n(x) = e^{-\Pi_2^n(0, \infty)} \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(qn)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+qn)}} \right]} + e^{\Pi_2^n(0, \infty)} \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+qn)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(qn)}} \right]}, \quad (4.38)$$

and

$$2 \int_{0^-}^\infty e^{-sx} dG_{21}^n(x) = e^{\Pi_2^n(0, \infty)} \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+qn)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(qn)}} \right]} - e^{-\Pi_2^n(0, \infty)} \frac{\mathbb{E} \left[ e^{-s\bar{X}_{e(qn)}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(p+qn)}} \right]}. \quad (4.39)$$

Obviously, we have

$$\lim_{n \uparrow \infty} \Pi_2^n(0, \infty) = \Pi_2(0, \infty) := \int_0^\infty \frac{1}{t} e^{-qt} (e^{-pt} - 1) \mathbb{P}(X_t > 0) dt. \quad (4.40)$$

Then from (4.29) and (4.32), we immediately derive that

$$\int_{0^-}^\infty e^{-sx} dG_{2i}^n(x) = \int_{0^-}^\infty e^{-sx} dG_{2i}(x), \quad i = 1, 2, \quad (4.41)$$

where  $G_{21}(x)$  and  $G_{22}(x)$  are given by (2.5), (2.6) and (2.7) with  $X$  in (4.1). Applying the continuity theorem for Laplace transforms again leads to

$$\lim_{n \uparrow \infty} G_{2i}^n(x) = G_{2i}(x), \quad \text{for } x > 0 \text{ and } i = 1, 2. \quad (4.42)$$

Recall (2.18) and (2.19). From (4.40), for  $i = 1, 2$ , it is easy to show that

$$\lim_{n \uparrow \infty} G_{2i}^n(0) = G_{2i}(0) \quad \text{and} \quad \lim_{n \uparrow \infty} G_{2i}^n(\infty) = G_{2i}(\infty) < \infty. \quad (4.43)$$

Since  $G_{21}(x)$  and  $G_{22}(x)$  are measures, it follows from (4.42) and (4.43) that  $G_{2i}^n(x)$  is uniformly convergence to  $G_{2i}(x)$  on  $[0, \infty]$ , where  $i = 1, 2$ .

Therefore, from (3.6) and (4.36), we arrive at the result that  $F_0^n(x)$  is uniformly convergence to  $F_1(x)$  on  $[0, \infty]$ . Since  $q_n > q$ , formula (3.9) gives us

$$|F_1(x)|, |F_0^n(x)| < e^{2 \int_0^\infty \frac{1}{t} e^{-qt} (e^{-pt} - 1) dt} + 1, \quad (4.44)$$

this completes the proof.  $\square$

**Proposition 4.3** *For  $X$  in (4.1), Theorem 3.1 is valid for the case of  $q \in \mathbb{Q}^c$  or  $p + q \in \mathbb{Q}^c$ .*

*Proof* From Lemma 4.5 (i), we know  $F_0^n(x)$  is uniformly convergence to  $F_1(x)$  on  $[0, \infty)$ . Thus if  $x_n \rightarrow x \geq 0$ , then

$$\lim_{n \uparrow \infty} F_0^n(x_n) = F_1(x). \quad (4.45)$$

For  $y \geq b$ ,  $J_0^n(x; y - b)$  can be written as (see (4.24))

$$J_0^n(x; y - b) = \mathbb{E} \left[ F_0^n(x - Z_0^n + y - b) \mathbf{1}_{\{Z_0^n < x\}} \right], \quad (4.46)$$

where the law of  $Z_0^n$  is given by  $K_{q_n}(z)$ . Besides, as Lemma 4.5 (ii) and Proposition 3.1 hold, we deduce from (4.27), (4.45) and (4.46) that

$$\lim_{n \uparrow \infty} J_0^n(x; y - b) = J_1(x; y - b) = \mathbb{E} \left[ F_1(x - Z_0 + y - b) \mathbf{1}_{\{Z_0 < x\}} \right], \quad (4.47)$$

where the distribution of  $Z_0$  is given by  $K_q(z)$ ; and we have used the bounded convergence theorem in the derivation of (4.47).

Therefore, the desired result is deduced by letting  $n \uparrow \infty$  in (4.23) and using (4.26) and (4.47).  $\square$

## 5 Proof of Theorem 3.1

In this section, the details on the derivation of Theorem 3.1 are given. The following technical lemma is important, and one can refer to Proposition 1 in Asmussen et al. [2] for its proof.

**Lemma 5.1** *For any given Lévy process  $X = (X_t)_{t \geq 0}$ , there exists a sequence of  $X^n = (X_t^n)_{t \geq 0}$  with the form of (4.1) such that*

$$\lim_{n \uparrow \infty} \sup_{s \in [0, t]} |X_s^n - X_s| = 0, \text{ almost surely.} \quad (5.1)$$

**Remark 5.1** *Note that  $X^n$  in (5.1) is assumed to have a Gaussian component. A particular case is that the process  $X$  in (5.1) is a pure jump process and at first sight Proposition 1 in Asmussen et al. [2] does not cover this special case. Here, for any given Lévy process  $X$ , we remind the reader that*

$$\lim_{n \uparrow \infty} \sup_{s \in [0, t]} \left| \left( X_s + \frac{1}{n} W_s \right) - X_s \right| = \lim_{n \uparrow \infty} \frac{1}{n} \overline{W}_T = 0,$$

where  $W_t$  is a Brownian motion. This means that Proposition 1 in Asmussen et al. [2] also holds for the above case.

For each  $X^n$  and  $y \geq b$ , Propositions 4.2 and 4.3 give

$$V_q^n(x) - \mathbb{P}_x(X_{e(q)}^n > y) = J_1^n(b - x; y - b), \quad (5.2)$$

with

$$V_q^n(x) := \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s^n \leq b\}} ds} \mathbf{1}_{\{X_{e(q)}^n > y\}} \right], \quad (5.3)$$

and

$$J_1^n(x; y - b) = \int_{-\infty}^x F_1^n(x - z + y - b) dK_q^n(z), \quad (5.4)$$

where  $K_q^n(x)$  is the convolution of  $\underline{X}_{e(q)}^n$  and  $\overline{X}_{e(p+q)}^n$  under  $\mathbb{P}$ , and

$$\int_0^\infty e^{-sx} F_1^n(x) dx = \frac{1}{s} \left( \frac{\mathbb{E} \left[ e^{-s\overline{X}_{e(q)}^n} \right]}{\mathbb{E} \left[ e^{-s\overline{X}_{e(p+q)}^n} \right]} - 1 \right), \quad s > 0. \quad (5.5)$$

**Lemma 5.2** *It holds that*

$$\lim_{n \uparrow \infty} V_q^n(x) = V_q(x), \quad \lim_{n \uparrow \infty} \mathbb{P}_x(X_{e(q)}^n > y) = \mathbb{P}_x(X_{e(q)} > y), \quad (5.6)$$

and

$$\lim_{n \uparrow \infty} K_q^n(x) = K_q(x), \quad x \in \mathbb{R}, \quad (5.7)$$

where  $K_q(x)$  is given by (3.4).

*Proof* Since Lemmas 2.1 and 5.1 hold, the dominated convergence theorem will lead to

$$\lim_{n \uparrow \infty} \mathbb{P}_x(X_{e(q)}^n > z) = \lim_{n \uparrow \infty} q \mathbb{E} \left[ \int_0^\infty e^{-qt} \mathbf{1}_{\{X_t^n > z\}} dt \right] = \mathbb{P}_x(X_{e(q)} > z). \quad (5.8)$$

Similarly, we have (note that  $p + q > 0$ )

$$\lim_{n \uparrow \infty} V_q^n(x) := \lim_{n \uparrow \infty} q \mathbb{E}_x \left[ \int_0^\infty e^{-qt} e^{-p \int_0^t \mathbf{1}_{\{X_s^n \leq b\}} ds} \mathbf{1}_{\{X_t^n > y\}} dt \right] = V_q(x). \quad (5.9)$$

In addition, it is known that (see, e.g., Lemma 13.4.1 of Whitt [17])

$$|\overline{X}_t^n - \overline{X}_t| \leq \sup_{0 \leq s \leq t} |X_s^n - X_s| \text{ and } |\underline{X}_t^n - \underline{X}_t| \leq \sup_{0 \leq s \leq t} |X_s^n - X_s|, \quad (5.10)$$

which combined with (5.1), yields

$$\lim_{n \uparrow \infty} \mathbb{E} \left[ e^{-s\overline{X}_{e(q)}^n} \right] = q \lim_{n \uparrow \infty} \int_0^\infty e^{-qt} \mathbb{E} \left[ e^{-s\overline{X}_t^n} \right] dt = \mathbb{E} \left[ e^{-s\overline{X}_{e(q)}} \right], \quad (5.11)$$

and

$$\lim_{n \uparrow \infty} \mathbb{E} \left[ e^{s\underline{X}_{e(q)}^n} \right] = q \lim_{n \uparrow \infty} \int_0^\infty e^{-qt} \mathbb{E} \left[ e^{s\underline{X}_t^n} \right] dt = \mathbb{E} \left[ e^{s\underline{X}_{e(q)}} \right], \quad (5.12)$$

where  $s, q > 0$ . Formulas (5.11) and (5.12) mean that  $\overline{X}_{e(p+q)}^n$  and  $\underline{X}_{e(q)}^n$  converge respectively to  $\overline{X}_{e(p+q)}$  and  $\underline{X}_{e(q)}$  in distribution, thus formula (5.7) is derived by recalling Proposition 3.1.  $\square$

**Lemma 5.3** For the continuous functions  $F_1(x)$  in (3.5) and  $F_1^n(x)$  in (5.5), the following results hold.

- (i)  $F_1^n(x)$  is uniformly convergence to  $F_1(x)$  on  $[0, \infty)$ .
- (ii)  $F_1(x), F_1^1(x), F_1^2(x), \dots$ , are uniformly bounded.

*Proof* Note that (5.11) holds for any  $q > 0$ . Besides, from (5.1), we have

$$\lim_{n \uparrow \infty} \int_0^\infty \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t^n > 0) dt = \int_0^\infty \frac{1}{t} e^{-qt} (1 - e^{-pt}) \mathbb{P}(X_t > 0) dt,$$

which is due to the result that  $\mathbb{P}(X_t = 0) = 0$  for Lebesgue almost every  $t > 0$  (see Lemma 2.1) and the dominated convergence theorem (note that  $p + q > 0$ ).

The remaining proof of this lemma is similar to that of Lemma 4.5, thus the details are omitted for simplicity.  $\square$

*Proof of Theorem 3.1* First, Lemma 5.3 (i) states that if  $x_n \rightarrow x \geq 0$ , then

$$\lim_{n \uparrow \infty} F_1^n(x_n) = F_1(x). \quad (5.13)$$

Next, for  $y \geq b$ ,  $J_1^n(x; y - b)$  in (5.4) can be written as

$$J_1^n(x; y - b) = \mathbb{E} \left[ F_1^n(x - Z_1^n + y - b) \mathbf{1}_{\{Z_1^n < x\}} \right], \quad (5.14)$$

where the law of  $Z_1^n$  is given by  $K_q^n(z)$ . Since  $F_1(x), F_1^1(x), F_1^2(x), \dots$ , are uniformly bounded (see Lemma 5.3 (ii)). Applying the bounded convergence theorem to (5.14) and using (5.7), (5.13) and Proposition 3.1, we obtain

$$\lim_{n \uparrow \infty} J_1^n(x; y - b) = J_1(x; y - b) = \mathbb{E} \left[ F_1(x - Z_1 + y - b) \mathbf{1}_{\{Z_1 < x\}} \right], \quad (5.15)$$

where the distribution of  $Z_1$  is given by  $K_q(z)$ . Therefore, letting  $n \uparrow \infty$  in (5.2), we derive Theorem 3.1 from (5.6) and (5.15).  $\square$

## 6 Examples

In Corollary 3.3 (see also Remark 3.10), we obtain expresses for the following expectation:

$$\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] = q \int_0^\infty e^{-qt} \mathbb{E}_x \left[ e^{-p \int_0^t \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_t \in dy\}} \right] dt, \quad (6.1)$$

where  $p, q > 0$  and  $X$  is a general Lévy process but not a compound Poisson process.

For some Lévy process  $X$ , this quantity  $\mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right]$  has more explicit expressions. And in the following, we will give some examples.

**Example 6.1** Let  $X$  be a hyper-exponential jump diffusion process, i.e., the process  $X$  is given by (4.1) but the jump distributions  $p^+(z)$  and  $p^-(z)$  are simplified as:

$$p^+(z) = \sum_{k=1}^{m^+} c_k \eta_k e^{-\eta_k z}, \quad p^-(z) = \sum_{k=1}^{n^-} d_k \vartheta_k e^{-\vartheta_k z}, \quad z > 0,$$

where  $c_k, \eta_k, \vartheta_k, d_k > 0$  and  $\sum_{k=1}^{m^+} c_k = 1 = \sum_{k=1}^{n^-} d_k$ .

In this case, the equation  $\psi(z) = q$  for any  $q > 0$  only has real and simple roots, where  $\psi(z) = \ln(\mathbb{E}[e^{zX_1}])$  (see Lemma 2.1 in [4]). The distributions of  $\bar{X}_{e(q)}$  and  $\underline{X}_{e(q)}$  for any  $q > 0$  have semi-explicit expressions, whose forms are the same as (A.2) and (A.5). In addition, the function  $F_1(x)$  defined by (3.5) has the same form as  $F_0(x)$  given by (4.12), and  $\hat{F}_2(x)$  in (3.28) has a similar expression. Thus the left-hand side of (3.31) can be written in a more explicit form.

In fact, after some simple calculations, we obtain that formula (3.31) will reduce to the results given by Corollary 3.9 in [18].  $\square$

**Example 6.2** Assume that  $X$  is a spectrally negative Lévy process. First, we give some results on such a Lévy process  $X$  and refer the reader to chapter 8 of [10] for more details.

It is known that for  $\lambda > 0$ ,

$$\psi(\lambda) = \ln(\mathbb{E}[e^{\lambda X_1}]) = \frac{1}{2}\sigma^2\lambda^2 + \gamma\lambda + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x \mathbf{I}_{\{x > -1\}}) \Pi(dx),$$

where  $\gamma \in \mathbb{R}$  and  $\sigma \geq 0$ ; the Lévy measure  $\Pi$  has a support of  $(-\infty, 0)$  such that  $\int_{-\infty}^0 (x^2 \wedge 1) \Pi(dx) < \infty$ . Besides, if  $\sigma = 0$  and  $\int_{-1}^0 |x| \Pi(dx) < \infty$ , then  $X$  has bounded variation and  $\psi(\lambda) = d\lambda + \int_{-\infty}^0 (e^{\lambda x} - 1) \Pi(dx)$  with  $d = \gamma - \int_{-1}^0 x \Pi(dx) > 0$ .

Define

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \quad \text{for } q > 0. \quad (6.2)$$

For given  $q > 0$ , the  $q$ -scale function  $W^{(q)}(x)$  is strictly increasing and continuous on  $(0, \infty)$  and its Laplace transform satisfies

$$\int_0^\infty e^{-sx} W^{(q)}(x) dx = \frac{1}{\psi(s) - q}, \quad \text{for } s > \Phi(q). \quad (6.3)$$

In addition,  $W^{(q)}(x) = 0$  for  $x < 0$  and  $W^{(q)}(0) := \lim_{x \downarrow 0} W^{(q)}(x)$ .

Next, we derive formulas for  $F_1(x)$ ,  $\hat{F}_2(x)$  and  $K_q(x)$  given respectively by (3.5), (3.28) and (3.4).

From (8.2) in [10], we know

$$\mathbb{E}[e^{-s\bar{X}_{e(q)}}] = \frac{\Phi(q)}{\Phi(q) + s} \quad \text{and} \quad \mathbb{E}[e^{s\underline{X}_{e(q)}}] = \frac{q}{\Phi(q)} \frac{\Phi(q) - s}{q - \psi(s)}, \quad s, q > 0. \quad (6.4)$$

(i) It follows directly from (3.13) and (6.4) that  $F_1(0) = \frac{\Phi(q)}{\Phi(p+q)} - 1$  and

$$F_1(dx) = \frac{\Phi(p+q) - \Phi(q)}{\Phi(p+q)} \Phi(q) e^{-\Phi(q)x} dx, \quad x > 0. \quad (6.5)$$

(ii) Note that (see (3.28))

$$\int_0^\infty e^{-sx} \hat{F}_2(dx) + \hat{F}_2(0) + 1 = \mathbb{E} \left[ e^{s\mathbf{X}_{e(p+q)}} \right] / \mathbb{E} \left[ e^{s\mathbf{X}_{e(q)}} \right].$$

So formula (6.4) gives  $\hat{F}_2(0) = \frac{(p+q)\Phi(q)}{q\Phi(p+q)} - 1$ . Then, for  $s > \max\{\Phi(p+q), \Phi(q)\}$ , some straightforward calculations will yield

$$\begin{aligned} \int_0^\infty e^{-sx} \hat{F}_2(dx) &= \mathbb{E} \left[ e^{s\mathbf{X}_{e(p+q)}} \right] / \mathbb{E} \left[ e^{s\mathbf{X}_{e(q)}} \right] - \frac{(p+q)\Phi(q)}{q\Phi(p+q)} \\ &= \frac{(p+q)\Phi(q)(\Phi(q) - \Phi(p+q))}{q\Phi(p+q)(s - \Phi(q))} + \frac{(p+q)p\Phi(q)}{q\Phi(p+q)} \frac{1 + \frac{\Phi(q) - \Phi(p+q)}{s - \Phi(q)}}{\psi(s) - (p+q)}. \end{aligned}$$

The above result and formula (6.3) will lead to

$$\frac{\hat{F}_2(dx)}{dx} = \frac{(p+q)\Phi(q)(\Phi(q) - \Phi(p+q))}{q\Phi(p+q)} e^{\Phi(q)x} + \frac{(p+q)p\Phi(q)}{q\Phi(p+q)} \hat{f}_2(x), \quad x > 0, \quad (6.6)$$

where

$$\hat{f}_2(x) = W^{(p+q)}(x) + (\Phi(q) - \Phi(p+q)) \int_0^x e^{\Phi(q)(x-z)} W^{(p+q)}(z) dz, \quad x \in \mathbb{R}. \quad (6.7)$$

(iii) Since  $\mathbb{P}(\bar{\mathbf{X}}_{e(p+q)} \in dx) = \Phi(p+q)e^{-\Phi(p+q)x}dx$  for  $x > 0$  (see (6.4)) and (see, e.g., formula (8.20) on page 219 of [10])

$$\mathbb{P}(-\mathbf{X}_{e(q)} \in dx) = \frac{q}{\Phi(q)} W^{(q)}(dx) - qW^{(q)}(x)dx, \quad x \geq 0.$$

From (3.4), we can rewrite  $K_q(dx)$  as

$$\Phi(p+q) \left\{ \int_{-\infty}^{0^+} e^{-\Phi(p+q)(x-z)} \mathbb{P}(\mathbf{X}_{e(q)} \in dz) - \int_x^{0^+} e^{-\Phi(p+q)(x-z)} \mathbb{P}(\mathbf{X}_{e(q)} \in dz) \right\}.$$

Integration by parts shows that

$$\int_x^{0^+} e^{-\Phi(p+q)(x-z)} W^{(q)}(-dz) = W^{(q)}(-x) + \Phi(p+q) \int_x^0 e^{-\Phi(p+q)(x-z)} W^{(q)}(-z) dz.$$

From (6.4) and the last three formulas, we will derive the following result after some simple computations.

$$\frac{K_q(dx)}{dx} = \frac{q}{p} \Phi(p+q) \left( \frac{\Phi(p+q)}{\Phi(q)} - 1 \right) e^{-\Phi(p+q)x} - \frac{q\Phi(p+q)}{\Phi(q)} k_q(x), \quad x \in \mathbb{R}, \quad (6.8)$$

where

$$k_q(x) = W^{(q)}(-x) + (\Phi(p+q) - \Phi(q)) e^{-\Phi(p+q)x} \int_x^0 e^{\Phi(p+q)z} W^{(q)}(-z) dz. \quad (6.9)$$



In addition, the definition of  $\hat{L}_q(x)$  (see Corollary 3.2 (ii)) gives

$$\hat{L}_q(dx) = K_q(-dx), \text{ for } x \in \mathbb{R},$$

which can be obtained from (6.8).

Finally, as  $\hat{F}_2(0) = \frac{(p+q)\Phi(q)}{q\Phi(p+q)} - 1$  and  $F_1(0) = \frac{\Phi(q)}{\Phi(p+q)} - 1$ , we can rewrite formula (3.33) as follows:

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] \\ &= \begin{cases} \frac{\Phi(q)}{\Phi(p+q)} K_q(dy - x) + \int_{b-x}^{y-x} F_1(dy - x - z) dK_q(z), & y \geq b, \\ \frac{q}{p+q} \left( \frac{(p+q)\Phi(q)}{q\Phi(p+q)} \hat{L}_q(x - dy) + \int_{x-b}^{x-y} \hat{F}_2(x - dy - z) d\hat{L}_q(z) \right), & y \leq b, \end{cases} \end{aligned} \quad (6.10)$$

which combined with (6.5), (6.6) and (6.8), leads to (see Appendix B for the details on the derivation)

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s < b\}} ds} \mathbf{1}_{\{X_{e(q)} \in dy\}} \right] = -q W_{x-b}^{(q,p)}(x-y) dy \\ & + \frac{q}{p} (\Phi(p+q) - \Phi(q)) H^{(p+q,-p)}(x-b) H^{(q,p)}(b-y) dy, \end{aligned} \quad (6.11)$$

where

$$\begin{aligned} H^{(p+q,-p)}(x-b) &= e^{\Phi(p+q)(x-b)} \left[ 1 - p \int_0^{x-b} e^{-\Phi(p+q)z} W^{(q)}(z) dz \right], \\ H^{(q,p)}(b-y) &= e^{\Phi(q)(b-y)} \left[ 1 + p \int_0^{b-y} e^{-\Phi(q)z} W^{(p+q)}(z) dz \right], \end{aligned} \quad (6.12)$$

and

$$W_{x-b}^{(q,p)}(x-y) = W^{(q)}(x-y) + p \int_{x-b}^{x-y} W^{(p+q)}(x-y-z) W^{(q)}(z) dz. \quad (6.13)$$

Therefore, formula (6.11) recovers the result obtained in previous research, see (19) in [11] or (12) in [19].

## 7 Conclusion

In this paper, we investigate the occupation times of a general Lévy process. Formulas for the Laplace transform of the joint distribution of an arbitrary Lévy process (which is not a compound Poisson process) and its occupation times are derived. The approach used is novel and the result has some applications in finance. Particularly, the application of our result to price occupation time derivatives is a potential direction of our future research.

## A The proof of Proposition 4.1

In this section, we derive Proposition 4.1 and present some preliminary results before starting the derivation.

The following Lemma A.1 gives the distributions of  $\bar{X}_{e(q)}$  and  $\underline{X}_{e(q)}$  for any  $q \in \mathbb{Q}$ , where Lemma A.1 (i) is taken from Theorem 2.2 and Corollary 2.1 in Lewis and Mordecki [14]; and Lemma A.1 (ii) is a straightforward application of Lemma A.1 (i) to the dual process  $-X$ .

**Lemma A.1** (i) For any  $q \in \mathbb{Q}$  and  $\text{Re}(s) \geq 0$ ,

$$\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right] = \prod_{k=1}^{m^+} \left( \frac{s + \eta_k}{\eta_k} \right)^{m_k} \prod_{k=1}^M \left( \frac{\beta_{k,q}}{s + \beta_{k,q}} \right) = \sum_{k=1}^M \frac{C_k^q}{s + \beta_{k,q}} := \psi_q^+(s), \quad (\text{A.1})$$

and for  $z \geq 0$ ,

$$\mathbb{P}(\bar{X}_{e(q)} \in dz) = \sum_{k=1}^M C_k^q e^{-\beta_{k,q}z} dz, \quad (\text{A.2})$$

where

$$\frac{C_i^q}{\beta_{i,q}} = \prod_{k=1}^{m^+} \left( \frac{\eta_k - \beta_{i,q}}{\eta_k} \right)^{m_k} \prod_{k=1, k \neq i}^M \frac{\beta_{k,q}}{\beta_{k,q} - \beta_{i,q}}, \quad \text{for } 1 \leq i \leq M. \quad (\text{A.3})$$

(ii) For any  $q \in \mathbb{Q}$  and  $\text{Re}(s) \geq 0$ ,

$$\mathbb{E} \left[ e^{s\underline{X}_{e(q)}} \right] = \prod_{k=1}^{n^-} \left( \frac{s + \vartheta_k}{\vartheta_k} \right)^{n_k} \prod_{k=1}^N \left( \frac{\gamma_{k,q}}{s + \gamma_{k,q}} \right) = \sum_{k=1}^N \frac{D_k^q}{s + \gamma_{k,q}} := \psi_q^-(s), \quad (\text{A.4})$$

and for  $z \leq 0$ ,

$$\mathbb{P}(\underline{X}_{e(q)} \in dz) = \sum_{k=1}^N D_k^q e^{\gamma_{k,q}z} dz, \quad (\text{A.5})$$

where

$$\frac{D_j^q}{\gamma_{j,q}} = \prod_{k=1}^{n^-} \left( \frac{\vartheta_k - \gamma_{j,q}}{\vartheta_k} \right)^{n_k} \prod_{k=1, k \neq j}^N \left( \frac{\gamma_{k,q}}{\gamma_{k,q} - \gamma_{j,q}} \right), \quad \text{for } 1 \leq j \leq N. \quad (\text{A.6})$$

Although (A.1) and (A.4) hold for  $\text{Re}(s) \geq 0$  only,  $\psi_q^+(s)$  in (A.1) and  $\psi_q^-(s)$  in (A.4) are treated as two rational functions of  $s$  in what follows. In addition, for any  $a \in \mathbb{R}$ , define

$$\tau_a^+ := \inf\{t \geq 0 : X_t > a\} \quad \text{and} \quad \tau_a^- := \inf\{t \geq 0 : X_t < a\}. \quad (\text{A.7})$$

Lemma A.2 summarizes the results on the one-sided exit problems of  $X$ , and its proof is very easy by applying Lemma A.1 and the following two results (see, Corollary 2 and formula (4) in Alili and Kyprianou [1])

$$\mathbb{E} \left[ e^{-q\tau_x^- + s(X_{\tau_x^-} - x)} \right] = \frac{\mathbb{E} \left[ e^{s(\underline{X}_{e(q)} - x)} \mathbf{1}_{\{\underline{X}_{e(q)} < x\}} \right]}{\mathbb{E} \left[ e^{s\underline{X}_{e(q)}} \right]}, \quad s \geq 0 \quad \text{and} \quad x \leq 0,$$

and

$$\mathbb{E} \left[ e^{-q\tau_x^+ - s(X_{\tau_x^+} - x)} \right] = \frac{\mathbb{E} \left[ e^{-s(\bar{X}_{e(q)} - x)} \mathbf{1}_{\{\bar{X}_{e(q)} > x\}} \right]}{\mathbb{E} \left[ e^{-s\bar{X}_{e(q)}} \right]}, \quad x, s \geq 0.$$

**Lemma A.2** (1) For  $q \in \mathbb{Q}$  and  $x, y \leq 0$ , we have

$$\mathbb{E} \left[ e^{-q\tau_x^-} \mathbf{I}_{\{X_{\tau_x^-} - x \in dy\}} \right] = D_0^q(x) \delta_0(dy) + \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} D_{kj}^q(x) \frac{(\vartheta_k)^j (-y)^{j-1}}{(j-1)!} e^{\vartheta_k y} dy, \quad (\text{A.8})$$

with  $D_0^q(x)$  and  $D_{kj}^q(x)$  given by rational expansion:

$$D_0^q(x) + \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} D_{kj}^q(x) \left( \frac{\vartheta_k}{\vartheta_k + s} \right)^j = \frac{1}{\psi_q^-(s)} \sum_{k=1}^N D_k^q \frac{e^{\gamma_{k,q}x}}{s + \gamma_{k,q}}, \quad x \leq 0, \quad (\text{A.9})$$

where  $\psi_q^-(s)$  is a rational function and is given by (A.4).

(2) For  $q \in \mathbb{Q}$  and  $x, y \geq 0$ ,

$$\mathbb{E} \left[ e^{-q\tau_x^+} \mathbf{1}_{\{X_{\tau_x^+} - x \in dy\}} \right] = C_0^q(x) \delta_0(dy) + \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} C_{kj}^q(x) \frac{(\eta_k)^j y^{j-1}}{(j-1)!} e^{-\eta_k y} dy, \quad (\text{A.10})$$

with  $C_0^q(x)$  and  $C_{kj}^q(x)$  given by rational expansion:

$$C_0^q(x) + \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} C_{kj}^q(x) \left( \frac{\eta_k}{\eta_k + s} \right)^j = \frac{1}{\psi_q^+(s)} \sum_{k=1}^M C_k^q \frac{e^{-\beta_{k,q}x}}{s + \beta_{k,q}}, \quad x \geq 0, \quad (\text{A.11})$$

where  $\psi_q^+(s)$  is a rational function and is given by (A.1).

**Remark A.1** From (A.9), we see that  $D_0^q(x)$  and  $D_{kj}^q(x)$ , for  $1 \leq k \leq n^-$  and  $1 \leq j \leq n_k$ , are linear combinations of  $e^{\gamma_{i,q}x}$  for  $1 \leq i \leq N$ . Similarly, formula (A.11) leads to that  $C_0^q(x)$  and  $C_{kj}^q(x)$ , for  $1 \leq k \leq m^+$  and  $1 \leq j \leq m_k$ , are linear combinations of  $e^{\beta_{i,q}x}$  for  $1 \leq i \leq M$ .

**Lemma A.3** For any  $\theta > 0$  and  $s \neq -\eta_1, \dots, -\eta_{m^+}$  with  $\theta \neq s$ ,

$$\int_0^\infty e^{-\theta x} C_0^q(x) dx + \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} \int_0^\infty e^{-\theta x} C_{kj}^q(x) dx \left( \frac{\eta_k}{\eta_k + s} \right)^j = \frac{1}{s - \theta} \left( \frac{\psi_q^+(\theta)}{\psi_q^+(s)} - 1 \right), \quad (\text{A.12})$$

and for any  $\theta > 0$  and  $s \neq -\vartheta_1, \dots, -\vartheta_{n^-}$  with  $\theta \neq s$ ,

$$\int_{-\infty}^0 e^{\theta x} D_0^q(x) dx + \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} \int_{-\infty}^0 e^{\theta x} D_{kj}^q(x) dx \left( \frac{\vartheta_k}{\vartheta_k + s} \right)^j = \frac{1}{s - \theta} \left( \frac{\psi_q^-(\theta)}{\psi_q^-(s)} - 1 \right). \quad (\text{A.13})$$

*Proof* These results can be obtained from (A.9) and (A.11) after some direct algebraic manipulations. Here, we only remind that

$$\int_0^\infty e^{-\theta x} \frac{e^{-\beta_{k,q}x}}{s + \beta_{k,q}} dx = \frac{1}{s - \theta} \left( \frac{1}{\theta + \beta_{k,q}} - \frac{1}{s + \beta_{k,q}} \right).$$

□

**Proof of Proposition 4.1** The derivation consists of three steps.

Step 1. For given  $y > b$ , considering the function defined in (3.1), we have

$$\begin{aligned} V_q(x) &= \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \right] \\ &\leq \begin{cases} 1, & \text{if } p \geq 0, \\ \mathbb{E}_x \left[ e^{-pe(q)} \right] = \frac{q}{p+q}, & \text{if } -q < p < 0. \end{cases} \end{aligned} \quad (\text{A.14})$$

For  $x < b$ , we can obtain from the strong Markov property of the process  $X$  and the lack of memory property of  $e(q)$  that

$$\begin{aligned}
 V_q(x) &= \mathbb{E}_x \left[ e^{-p \int_0^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \mathbf{1}_{\{e(q) > \tau_b^+\}} \right] \\
 &= \mathbb{E}_x \left[ e^{-p \tau_b^+ - p \int_{\tau_b^+}^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \mathbf{1}_{\{e(q) > \tau_b^+\}} \right] \\
 &= \mathbb{E}_x \left[ e^{-(p+q)\tau_b^+} V_q(X_{\tau_b^+}) \right] \\
 &= \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} C_{kj}^\xi (b-x) \int_0^\infty \frac{(\eta_k)^j z^{j-1}}{(j-1)!} e^{-\eta_k z} V_q(b+z) dz \\
 &\quad + C_0^\xi (b-x) V_q(b) = \sum_{k=1}^M U_k e^{\beta_{k,\xi}(x-b)}, \quad x < b,
 \end{aligned} \tag{A.15}$$

where  $\xi = p + q$  and  $U_1, \dots, U_M$  are proper constants and do not depend on  $x$ ; the fourth equality follows from (A.10) and the final one is due to Remark A.1.

Similarly, for  $x > b$ , we can derive

$$\begin{aligned}
 V_q(x) &= \mathbb{E}_x \left[ e^{-p \int_{\tau_b^-}^{e(q)} \mathbf{1}_{\{X_s \leq b\}} ds} \mathbf{1}_{\{X_{e(q)} > y\}} \mathbf{1}_{\{e(q) > \tau_b^-\}} \right] \\
 &\quad + \mathbb{E}_x \left[ \mathbf{1}_{\{X_{e(q)} > y\}} \mathbf{1}_{\{e(q) \leq \tau_b^-\}} \right] \\
 &= \mathbb{E}_x \left[ e^{-q \tau_b^-} V_q(X_{\tau_b^-}) \right] + \mathbb{P}_x(X_{e(q)} > y, \underline{X}_{e(q)} \geq b) \\
 &= \sum_{k=1}^{n^-} \sum_{j=1}^{n_k} D_{kj}^q (b-x) \int_{-\infty}^0 V_q(b+z) \frac{(\vartheta_k)^j (-z)^{j-1}}{(j-1)!} e^{\vartheta_k z} dz \\
 &\quad + D_0^q (b-x) V_q(b) + \mathbb{P}_x(X_{e(q)} > y, \underline{X}_{e(q)} \geq b),
 \end{aligned} \tag{A.16}$$

where the third equality follows from (A.8).

The well-known Wiener-Hopf factorization (see, e.g., Theorem 6.16 in [10]) gives that  $X_{e(q)} - \underline{X}_{e(q)}$  is independent of  $\underline{X}_{e(q)}$  and is equal in distribution to  $\overline{X}_{e(q)}$  under  $\mathbb{P}$ . This result and formulas (A.2) and (A.5) yield

$$\begin{aligned}
 \mathbb{P}_x(X_{e(q)} > y, \underline{X}_{e(q)} \geq b) &= \int_{b-x}^0 \mathbb{P}(X_{e(q)} - \underline{X}_{e(q)} > y - x - z, \underline{X}_{e(q)} \in dz) \\
 &= \int_{b-x}^0 \mathbb{P}(\overline{X}_{e(q)} > y - x - z) \mathbb{P}(\underline{X}_{e(q)} \in dz) \\
 &= \begin{cases} \sum_{k=1}^M H_k e^{\beta_{k,q}(x-y)} + \sum_{k=1}^N \hat{P}_k e^{\gamma_{k,q}(b-x)}, & b < x \leq y, \\ 1 + \sum_{k=1}^N Q_k e^{\gamma_{k,q}(y-x)} + \sum_{k=1}^M \hat{P}_k e^{\gamma_{k,q}(b-x)}, & x \geq y, \end{cases}
 \end{aligned} \tag{A.17}$$

where we remind the reader that  $\mathbb{P}(\overline{X}_{e(q)} > z) = 1$  for  $z \leq 0$ ; for  $k = 1, 2, \dots, M$ ,

$$H_k = \frac{C_k^q}{\beta_{k,q}} \sum_{j=1}^N \frac{D_j^q}{\beta_{k,q} + \gamma_{j,q}}, \tag{A.18}$$

and for  $k = 1, 2, \dots, N$ ,

$$Q_k = D_k^q \sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}(\beta_{i,q} + \gamma_{k,q})} - \frac{D_k^q}{\gamma_{k,q}}, \quad \hat{P}_k = - \sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}} \frac{D_i^q e^{\beta_{i,q}(b-y)}}{\beta_{i,q} + \gamma_{k,q}}. \tag{A.19}$$

From (A.16), (A.17) and Remark A.1, we arrive at

$$V_q(x) = \begin{cases} \sum_{k=1}^M H_k e^{\beta_{k,q}(x-y)} + \sum_{k=1}^N P_k e^{\gamma_{k,q}(b-x)}, & b < x \leq y, \\ 1 + \sum_{k=1}^N Q_k e^{\gamma_{k,q}(y-x)} + \sum_{k=1}^M P_k e^{\gamma_{k,q}(b-x)}, & x \geq y, \end{cases} \tag{A.20}$$

where  $P_1, \dots, P_N$  do not depend on  $x$  and satisfy

$$\begin{aligned} \sum_{k=1}^N P_k e^{\gamma_{k,q}(b-x)} &= - \sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}} \sum_{j=1}^N \frac{D_j^q e^{\gamma_{j,q}(b-x)}}{\beta_{i,q} + \gamma_{j,q}} e^{\beta_{i,q}(b-y)} + D_0^q(b-x) V_q(b) \\ &+ \sum_{k=1}^n \sum_{j=1}^{n_k} D_{kj}^q(b-x) \int_{-\infty}^0 V_q(b+z) \frac{(\partial_k)^j (-z)^{j-1}}{(j-1)!} e^{\partial_k z} dz, \quad x > b. \end{aligned} \quad (\text{A.21})$$

Formulas (A.15) and (A.20) imply that the remaining thing is to derive the expressions of  $U_k$  and  $P_k$ . In the second step, we establish the equations satisfied by  $U_1, \dots, U_M$  and  $P_1, \dots, P_N$ ; and in the last step, we solve these equations.

Step 2. Since  $\sigma > 0$ , it is known that  $\mathbb{P}(\tau_0^+ = 0) = \mathbb{P}(\tau_0^- = 0) = 1$ . Thus  $V_q(x)$  is continuous at  $b$  (see (A.15) and (A.16)), i.e.,  $\lim_{x \uparrow b} V_q(x) = V_q(b) = \lim_{x \downarrow b} V_q(x)$ , which combined with (A.15) and (A.20), yields

$$\sum_{i=1}^M U_i = V_q(b) = \sum_{i=1}^M H_i e^{\beta_{i,q}(b-y)} + \sum_{i=1}^N P_i. \quad (\text{A.22})$$

Besides, we know that the derivative of  $V_q(x)$  at  $b$  is continuous, i.e.,  $V_q'(b-) = V_q'(b+)^2$ . Then, it follows from (A.15) and (A.20) that

$$\sum_{i=1}^M U_i \beta_{i,q} \xi = \sum_{i=1}^M H_i \beta_{i,q} e^{\beta_{i,q}(x-y)} - \sum_{i=1}^N P_i \gamma_{i,q}. \quad (\text{A.23})$$

For all  $\theta \in \mathbb{C}$  except at  $\beta_{1,q}, \dots, \beta_{M,q}$  and  $-\gamma_{1,q}, \dots, -\gamma_{N,q}$ , formulas (A.18), (A.19) and some straightforward computations lead to

$$\begin{aligned} \sum_{k=1}^M \frac{\theta H_k}{\beta_{k,q} - \theta} + 1 + \sum_{k=1}^N \frac{\theta Q_k}{\theta + \gamma_{k,q}} &= \sum_{k=1}^M \frac{\theta H_k}{\beta_{k,q} - \theta} + \sum_{i=1}^M \sum_{j=1}^N \frac{C_i^q D_j^q}{\beta_{i,q} \gamma_{j,q}} \\ &+ \sum_{k=1}^N \frac{\theta}{\theta + \gamma_{k,q}} \left( D_k^q \sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}(\beta_{i,q} + \gamma_{k,q})} - \frac{D_k^q}{\gamma_{k,q}} \sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}} \right) \\ &= \sum_{i=1}^M \sum_{j=1}^N \left\{ \frac{\theta C_i^q D_j^q}{\beta_{i,q}(\beta_{i,q} + \gamma_{j,q})(\beta_{i,q} - \theta)} + \frac{C_i^q D_j^q}{\beta_{i,q} \gamma_{j,q}} - \frac{\theta C_i^q D_j^q}{\gamma_{j,q}(\beta_{i,q} + \gamma_{j,q})(\gamma_{j,q} + \theta)} \right\} \\ &= \sum_{i=1}^M \sum_{j=1}^N \frac{C_i^q D_j^q}{(\beta_{i,q} - \theta)(\theta + \gamma_{j,q})} = \psi_q^+(-\theta) \psi_q^-(\theta), \end{aligned} \quad (\text{A.24})$$

where we have used  $\sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}} = 1 = \sum_{j=1}^N \frac{D_j^q}{\gamma_{j,q}}$  (let  $s = 0$  in (A.1) and (A.4)) in the first equality; the last equality follows from (A.1) and (A.4).

For  $1 \leq k \leq m^+$  and  $1 \leq j \leq m_k$ , we can write

$$(-1)^{j-1} \int_{b_1}^{b_2} z^{j-1} e^{-\eta_k z} e^{\theta z} dz = \frac{\partial^{j-1}}{\partial \eta^{j-1}} \left( \int_{b_1}^{b_2} e^{-\eta z} e^{\theta z} dz \right)_{\eta=\eta_k},$$

providing the integral  $\int_{b_1}^{b_2} e^{-\eta z} e^{\theta z} dz$  exists. So we derive via (A.20) that

$$\int_0^\infty \frac{(\eta_k)^j z^{j-1}}{(j-1)!} e^{-\eta_k z} V_q(b+z) dz = \sum_{i=1}^N \frac{P_i(\eta_k)^j}{(\eta_k + \gamma_{i,q})^j} + \sum_{i=1}^M \frac{H_i(\eta_k)^j}{(\eta_k - \beta_{i,q})^j} e^{\beta_{i,q}(b-y)}, \quad (\text{A.25})$$

where we have used the following result:

$$\frac{(\eta_k)^j (-1)^{j-1}}{(j-1)!} \frac{\partial^{j-1}}{\partial \eta^{j-1}} \left( \frac{1}{\eta} e^{\eta(b-y)} \left( \sum_{i=1}^M \frac{H_i \eta}{\beta_{i,q} - \eta} + \sum_{i=1}^N \frac{\eta Q_i}{\eta + \gamma_{i,q}} + 1 \right) \right)_{\eta=\eta_k} = 0, \quad (\text{A.26})$$

<sup>2</sup> From the proof given in the Appendix A of Wu and Zhou [18], we obtain that  $V_q'(b-)$  and  $V_q'(b+)$  must be equal if they are existent. The existence of  $V_q'(b-)$  and  $V_q'(b+)$  can be seen from (A.15) and (A.20).

which can be proved by using (A.24) and noting that

$$\frac{\partial^{j-1}}{\partial \eta^{j-1}} (\psi_q^+(-\eta))_{\eta=\eta_k} = 0, \text{ for } 1 \leq k \leq m^+ \text{ and } 1 \leq j \leq m_k. \quad (\text{A.27})$$

Then, it follows from (A.15), (A.22) and (A.25) that

$$\begin{aligned} \sum_{k=1}^M U_k e^{\beta_{k,\xi}(x-b)} &= C_0^\xi(b-x) \left( \sum_{i=1}^M H_i e^{\beta_{i,q}(b-y)} + \sum_{i=1}^N P_i \right) \\ &+ \sum_{k=1}^{m^+} \sum_{j=1}^{m_k} C_{kj}^\xi(b-x) \left( \sum_{i=1}^N \frac{P_i(\eta_k)^j}{(\eta_k + \gamma_{i,q})^j} + \sum_{i=1}^M \frac{H_i(\eta_k)^j}{(\eta_k - \beta_{i,q})^j} e^{\beta_{i,q}(b-y)} \right). \end{aligned} \quad (\text{A.28})$$

Multiplying both sides of (A.28) by  $e^{\theta(x-b)}$  and taking an integration from  $-\infty$  to  $b$  with respect to  $x$ , we obtain from (A.12) that

$$\begin{aligned} \sum_{i=1}^M \frac{U_i}{\beta_{i,\xi} + \theta} &= \sum_{i=1}^M \frac{H_i e^{\beta_{i,q}(b-y)}}{\theta + \beta_{i,q}} \left( 1 - \frac{\psi_\xi^+(\theta)}{\psi_\xi^+(-\beta_{i,q})} \right) \\ &+ \sum_{i=1}^N \frac{P_i}{\gamma_{i,q} - \theta} \left( \frac{\psi_\xi^+(\theta)}{\psi_\xi^+(\gamma_{i,q})} - 1 \right). \end{aligned} \quad (\text{A.29})$$

Since both sides of (A.29) are rational functions of  $\theta$ , it can be extended to the whole plane except at  $-\beta_{1,\xi}, \dots, -\beta_{M,\xi}$ . Note that

$$\lim_{\theta \rightarrow -\beta_{i,q}} \frac{\psi_\xi^+(-\beta_{i,q}) - \psi_\xi^+(\theta)}{\theta + \beta_{i,q}} = -\psi_\xi^{+'}(-\beta_{i,q}), \quad \lim_{\theta \rightarrow \gamma_{i,q}} \frac{\psi_\xi^+(\theta) - \psi_\xi^+(\gamma_{i,q})}{\theta - \gamma_{i,q}} = \psi_\xi^{+'}(\gamma_{i,q}).$$

Similarly, for  $1 \leq k \leq n^-$  and  $1 \leq j \leq n_k$ , we can derive from (A.15) that

$$\int_{-\infty}^0 V_q(b+z) \frac{(\vartheta_k)^j (-z)^{j-1}}{(j-1)!} e^{\vartheta_k z} dz = \sum_{i=1}^M \frac{U_i (\vartheta_k)^j}{(\vartheta_k + \beta_{i,\xi})^j}. \quad (\text{A.30})$$

From (A.13), (A.21), (A.30) and the fact of  $V_q(b) = \sum_{i=1}^M U_i$  (see (A.22)), it can be shown that

$$\begin{aligned} \sum_{i=1}^N \frac{P_i}{\theta + \gamma_{i,q}} &= \sum_{i=1}^N P_i \int_b^\infty e^{\theta(b-x)} e^{\gamma_{i,q}(b-x)} dx \\ &= - \sum_{i=1}^M \sum_{j=1}^N \frac{D_j^q}{\beta_{i,q} + \gamma_{j,q}} \frac{e^{\beta_{i,q}(b-y)}}{\theta + \gamma_{j,q}} \frac{C_i^q}{\beta_{i,q}} + \sum_{i=1}^M \frac{U_i}{\beta_{i,\xi} - \theta} \left( \frac{\psi_q^-(\theta)}{\psi_q^-(\beta_{i,\xi})} - 1 \right). \end{aligned} \quad (\text{A.31})$$

Furthermore, it holds that

$$\begin{aligned} &- \sum_{i=1}^M \sum_{j=1}^N \frac{D_j^q}{\beta_{i,q} + \gamma_{j,q}} \frac{e^{\beta_{i,q}(b-y)}}{\theta + \gamma_{j,q}} \frac{C_i^q}{\beta_{i,q}} \\ &= - \sum_{i=1}^M \frac{C_i^q}{\beta_{i,q}} e^{\beta_{i,q}(b-y)} \frac{1}{\beta_{i,q} - \theta} \sum_{j=1}^N D_j^q \left( \frac{1}{\theta + \gamma_{j,q}} - \frac{1}{\beta_{i,q} + \gamma_{j,q}} \right) \\ &= \sum_{i=1}^M \frac{H_i}{\beta_{i,q} - \theta} e^{\beta_{i,q}(b-y)} - \sum_{i=1}^M \frac{C_i^q \psi_q^-(\theta)}{\beta_{i,q}(\beta_{i,q} - \theta)} e^{\beta_{i,q}(b-y)}. \end{aligned} \quad (\text{A.32})$$

where the second equality follows from (A.18) and (A.4). Hence,

$$\begin{aligned} \sum_{i=1}^N \frac{P_i}{\theta + \gamma_{i,q}} &= \sum_{i=1}^M \frac{U_i}{\beta_{i,\xi} - \theta} \left( \frac{\psi_q^-(\theta)}{\psi_q^-(\beta_{i,\xi})} - 1 \right) \\ &+ \sum_{i=1}^M \frac{H_i}{\beta_{i,q} - \theta} e^{\beta_{i,q}(b-y)} - \sum_{i=1}^M \frac{C_i^q \psi_q^-(\theta)}{\beta_{i,q}(\beta_{i,q} - \theta)} e^{\beta_{i,q}(b-y)}, \end{aligned} \quad (\text{A.33})$$

which holds for  $\theta \in \mathbb{C}$  except at  $-\gamma_{1,q}, \dots, -\gamma_{N,q}$ .

Therefore, for any given  $1 \leq k \leq m^+$  and  $0 \leq j \leq m_k - 1$ , taking a derivative on both sides of (A.29) with respect to  $\theta$  up to  $j$  order and letting  $\theta$  equal to  $-\eta_k$  will produce

$$\sum_{i=1}^M \frac{U_i(-1)^j}{(\beta_{i,\xi} - \eta_k)^{j+1}} + \sum_{i=1}^M \frac{P_i}{(\eta_k + \gamma_{i,q})^{j+1}} - \sum_{i=1}^M \frac{H_i(-1)^j}{(\beta_{i,q} - \eta_k)^{j+1}} e^{\beta_{i,q}(b-y)} = 0, \quad (\text{A.34})$$

where the fact that  $\frac{\partial^j}{\partial \theta^j} (\psi_\xi^+(\theta))_{\theta=-\eta_k} = 0$  (see (A.1)) is used in the derivation. In a similar way, for any given  $1 \leq k \leq n^-$  and  $0 \leq j \leq n_k - 1$ , applying  $\frac{\partial^j}{\partial \theta^j} (\psi_q^-(\theta))_{\theta=-\vartheta_k} = 0$  (see (A.4)) to (A.33) yields

$$\sum_{i=1}^M \frac{U_i(-1)^j}{(\beta_{i,\xi} + \theta_k)^{j+1}} + \sum_{i=1}^M \frac{P_i}{(\gamma_{i,q} - \theta_k)^{j+1}} - \sum_{i=1}^M \frac{H_i(-1)^j}{(\beta_{i,q} + \theta_k)^{j+1}} e^{\beta_{i,q}(b-y)} = 0. \quad (\text{A.35})$$

Step 3. Consider the following rational function of  $x$ :

$$f(x) = \sum_{i=1}^M \frac{U_i}{x - \beta_{i,\xi}} - \sum_{i=1}^N \frac{P_i}{x + \gamma_{i,q}} - \sum_{i=1}^M \frac{H_i}{x - \beta_{i,q}} e^{\beta_{i,q}(b-y)}. \quad (\text{A.36})$$

For fixed  $1 \leq k \leq m^+$  and  $0 \leq j \leq m_k - 1$ , (A.34) yields that  $\frac{\partial^j}{\partial x^j} (f(x))_{x=\eta_k} = 0$ , which means that  $\eta_k$  is a root of the equation  $f(x) = 0$  with multiplicity  $m_k$ . Besides, from (A.35), for  $1 \leq k \leq n^-$ , we obtain that  $-\vartheta_k$  is a  $n_k$ -multiplicity root of  $f(x) = 0$ . These results give us (recall  $M = \sum_{k=1}^{m^+} m_k + 1$  and  $N = \sum_{k=1}^{n^-} n_k + 1$ ; see Lemma 4.1)

$$f(x) = \frac{\prod_{k=1}^{m^+} (x - \eta_k)^{m_k} \prod_{k=1}^{n^-} (x + \vartheta_k)^{n_k} (l_0 + l_1 x + \dots + l_{M+1} x^{M+1})}{\prod_{i=1}^M (x - \beta_{i,\xi}) \prod_{i=1}^N (x + \gamma_{i,q}) \prod_{i=1}^M (x - \beta_{i,q})}, \quad (\text{A.37})$$

with some proper constants  $l_0, l_1, \dots, l_{M+1}$ .

Formulas (A.36) and (A.37) produce

$$l_{M+1} = \sum_{i=1}^M U_i - \sum_{i=1}^N P_i - \sum_{i=1}^M H_i e^{\beta_{i,q}(b-y)},$$

and

$$l_M = \sum_{i=1}^M U_i (Sum + \beta_{i,\xi}) - \sum_{i=1}^N P_i (Sum - \gamma_{i,q}) - \sum_{i=1}^M H_i e^{\beta_{i,q}(b-y)} (Sum + \beta_{i,q}),$$

where  $Sum = -\sum_{i=1}^M \beta_{i,\xi} + \sum_{i=1}^N \gamma_{i,q} - \sum_{i=1}^M \beta_{i,q}$ . So formulas (A.22) and (A.23) lead to that  $l_{M+1} = 0$  and  $l_M = 0$ . In addition, it is obvious that (see (A.36))

$$\lim_{x \rightarrow \beta_{i,q}} f(x)(x - \beta_{i,q}) = -H_i e^{\beta_{i,q}(b-y)}, \quad 1 \leq i \leq M, \quad (\text{A.38})$$

thus

$$\begin{aligned} f(x) &= \frac{\prod_{k=1}^{m^+} (x - \eta_k)^{m_k} \prod_{k=1}^{n^-} (x + \vartheta_k)^{n_k}}{\prod_{i=1}^M (x - \beta_{i,\xi}) \prod_{i=1}^N (x + \gamma_{i,q})} \times \\ &\sum_{k=1}^M \frac{\prod_{i=1}^M (\beta_{k,q} - \beta_{i,\xi}) \prod_{i=1}^N (\beta_{k,q} + \gamma_{i,q})}{\prod_{i=1}^{m^+} (\beta_{k,q} - \eta_i)^{m_i} \prod_{i=1}^{n^-} (\beta_{k,q} + \vartheta_i)^{n_i}} \frac{-H_k}{x - \beta_{k,q}} e^{\beta_{k,q}(b-y)}. \end{aligned} \quad (\text{A.39})$$

Formulas (4.7) and (4.8) are derived from (A.15), (A.20), (A.36) and (A.39).  $\square$

## B The derivation of (6.11)

In this section, we give the details on the derivation of (6.11), and we will divide the arguments into two cases.

**Case 1:** Assume  $y \geq b$ .

In this case, we have (see (6.12) and (6.13))

$$W_{x-b}^{(q,p)}(x-y) = W^{(q)}(x-y) \text{ and } H^{(q,p)}(b-y) = e^{\Phi(q)(b-y)}.$$

For any given  $x < 0$ , exchanging the order of integration produces

$$\int_x^0 e^{-\Phi(q)(x-z)} k_q(z) dz = e^{-\Phi(p+q)x} \int_x^0 e^{\Phi(p+q)z} W^{(q)}(-z) dz. \quad (\text{B.1})$$

Since  $k_q(z) = W^{(q)}(-z) = 0$  if  $z > 0$ . So identity (B.1) holds for all  $x \in \mathbb{R}$ . In addition, for any given  $x \in \mathbb{R}$ , we have

$$\begin{aligned} \int_{b-x}^{y-x} e^{-\Phi(q)(y-x-z)} k_q(z) dz &= e^{\Phi(q)(b-y)} \int_{b-x}^0 e^{-\Phi(q)(b-x-z)} k_q(z) dz \\ &\quad - \int_{y-x}^0 e^{-\Phi(q)(y-x-z)} k_q(z) dz. \end{aligned}$$

When  $y \geq b$ , applying (6.5), (6.8), (6.10) and the above three formulas, we can deduce (6.11) after some simple and straightforward computations.

**Case 2:** Assume  $y \leq b$ .

As  $\hat{L}_q(dx) = K_q(-dx)$ , it follows from (6.8) that

$$\frac{\hat{L}_q(dx)}{dx} = \frac{q}{p} \Phi(p+q) \left( \frac{\Phi(p+q)}{\Phi(q)} - 1 \right) e^{\Phi(p+q)x} - \frac{q\Phi(p+q)}{\Phi(q)} k_q(-x),$$

where  $k_q(-x)$  is given by (6.9).

Formula (B.1) can be rewritten as

$$\int_0^x e^{\Phi(q)(x-z)} k_q(-z) dz = e^{\Phi(p+q)x} \int_0^x e^{-\Phi(p+q)z} W^{(q)}(z) dz, \text{ for all } x \in \mathbb{R}.$$

Besides, for  $x \in \mathbb{R}$ , it is obvious that

$$\begin{aligned} \int_{x-b}^{x-y} e^{\Phi(q)(x-y-z)} k_q(-z) dz &= \int_0^{x-y} e^{\Phi(q)(x-y-z)} k_q(-z) dz \\ &\quad - e^{\Phi(q)(b-y)} \int_0^{x-b} e^{\Phi(q)(x-b-z)} k_q(-z) dz. \end{aligned}$$

Similar to the derivation of (B.1), from (6.7), we can obtain

$$\int_0^x e^{\Phi(p+q)(x-z)} \hat{f}_2(z) dz = e^{\Phi(q)x} \int_0^x e^{-\Phi(q)z} W^{(p+q)}(z) dz,$$

which holds for all  $x \in \mathbb{R}$  (note that  $\hat{f}_2(z) = W^{(p+q)}(z) = 0$  for  $z < 0$ ).

Finally, for all  $x \in \mathbb{R}$ , we will prove the following result in Lemma B.1.

$$\begin{aligned} \int_{x-b}^{x-y} \hat{f}_2(x-y-z) k_q(-z) dz &= \int_{x-b}^{x-y} W^{(p+q)}(x-y-z) W^{(q)}(z) dz \\ &\quad + (\Phi(p+q) - \Phi(q)) \int_0^{x-b} e^{\Phi(p+q)(x-b-z)} W^{(q)}(z) dz \int_0^{b-y} e^{\Phi(q)(b-y-z)} W^{(p+q)}(z) dz. \end{aligned} \quad (\text{B.2})$$

For given  $y \leq b$ , applying the above results, (6.6) and (6.10) will derive (6.11). This derivation only involves some straightforward calculations, thus we omit the details for brevity.

**Lemma B.1** For given  $y \leq b$ , formula (B.2) holds for all  $x \in \mathbb{R}$ .



*Proof* For  $s > \max\{\Phi(p+q), \Phi(q)\}$ , we can derive via (6.3), (6.7) and (6.9) that

$$\int_0^\infty e^{-sx} \int_0^x \hat{f}_2(x-z) k_q(-z) dz dx = \frac{1}{\Psi(s)-q} \frac{1}{\Psi(s)-(p+q)},$$

which leads to (note that  $k_q(-z) = W^{(q)}(z) = 0$  if  $z < 0$ )

$$\int_0^x \hat{f}_2(x-z) k_q(-z) dz = \int_0^x W^{(q)}(z) W^{(p+q)}(x-z) dz, \text{ for all } x \in \mathbb{R}. \quad (\text{B.3})$$

So formula (B.2) holds for  $x \leq b$  (note that the second item on the right-hand side of (B.2) equals to zero if  $x \leq b$ ).

If  $x > b$ , we note first that

$$\begin{aligned} \int_{x-b}^{x-y} dz \int_0^{x-y-z} dt_1 \int_{-z}^0 \hat{w}(\cdot) dt_2 &= \int_0^{b-y} dt_1 \int_{b-x}^0 dt_2 \int_{x-b}^{x-y-t_1} \hat{w}(\cdot) dz \\ &+ \int_0^{b-y} dt_1 \int_{t_1+y-x}^{b-x} dt_2 \int_{-t_2}^{x-y-t_1} \hat{w}(\cdot) dz, \end{aligned} \quad (\text{B.4})$$

where

$$\hat{w}(\cdot) = e^{\Phi(q)(x-y-z-t_1)} W^{(p+q)}(t_1) e^{\Phi(p+q)(z+t_2)} W^{(q)}(-t_2).$$

Then, from (6.7), (6.9) and (B.4), we can show that (B.2) also holds for  $x > b$  after some straightforward computations.  $\square$

**Remark B.1** For any given  $c \geq 0$  and  $x > 0$ , the following identity holds

$$\begin{aligned} \int_0^x \hat{f}_2(x+c-z) k_q(-z) dz &= \int_0^x W^{(p+q)}(x+c-z) W^{(q)}(z) dz \\ &- (\Phi(p+q) - \Phi(q)) \int_0^x e^{\Phi(p+q)(x-z)} W^{(q)}(z) dz \int_0^c e^{\Phi(q)(c-z)} W^{(p+q)}(z) dz, \end{aligned} \quad (\text{B.5})$$

since both sides have the same Laplace transform.

In fact, for  $s > \max\{\Phi(p+q), \Phi(q)\}$ , exchanging the order of integration gives

$$\begin{aligned} \int_0^\infty e^{-sx} \int_0^x \hat{f}_2(x+c-z) k_q(-z) dz dx &= \int_0^\infty \int_z^\infty e^{-sx} \hat{f}_2(x+c-z) dx k_q(-z) dz \\ &= \int_0^\infty e^{-sz} k_q(-z) dz \left( e^{sc} \int_0^\infty e^{-sx} \hat{f}_2(x) dx - \int_0^c e^{-s(x-c)} \hat{f}_2(x) dx \right), \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty e^{-sx} \int_0^x W^{(p+q)}(x+c-z) W^{(q)}(z) dz dx \\ = \int_0^\infty e^{-sz} W^{(q)}(z) dz \left( e^{sc} \int_0^\infty e^{-sx} W^{(p+q)}(x) dx - \int_0^c e^{-s(x-c)} W^{(p+q)}(x) dx \right). \end{aligned}$$

Note that

$$\int_0^c e^{-s(x-c)} \int_0^x e^{\Phi(q)(x-z)} W^{(p+q)}(z) dz dx = \int_0^c \int_z^c e^{-s(x-c)} e^{\Phi(q)(x-z)} dx W^{(p+q)}(z) dz.$$

Combining the above three formulas with (6.3), (6.7) and (6.9), we can deduce that both sides of (B.5) have the same Laplace transform.  $\square$

**Remark B.2** Note first that identity (B.5) holds also for  $x \leq 0$  (since  $k_q(-z) = W^{(q)}(z) = 0$  if  $z < 0$ ). For any given  $x \in \mathbb{R}$  and  $y \leq b$ , we can write

$$\begin{aligned} \int_{x-b}^{x-y} \hat{f}_2(x-y-z) k_q(-z) dz &= \int_0^{x-y} \hat{f}_2(x-y-z) k_q(-z) dz \\ &- \int_0^{x-b} \hat{f}_2(x-b+(b-y)-z) k_q(-z) dz. \end{aligned}$$

Then formula (B.2) follows from (B.5).  $\square$

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